Initial Data Optimization Problem for One Class of Neutral Functional Differential Equation with the Continuous Initial Condition

Tamaz Tadumadze

I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: tamaz.tadumadze@tsu.ge

Nika Gorgodze

A. Tsereteli State University, Kutaisi, Georgia
E-mail: nika_gorgodze@yahoo.com

Let $I = [a, b]$ be a finite interval and let $\mathbb{R}^n$ be an n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T means transposition. Suppose that $O \subset \mathbb{R}^n$ is an open set and $M \subset O$ is a convex set. Let the function $f(t, x, y) = (f^1(t, x, y), \dots, f^n(t, x, y))^T$ be defined on $I \times O^2$ and satisfy the following conditions: for almost all fixed $t \in I$ the function $f(t, x, y)$ is continuously differentiable with respect to $(x, y) \in O^2$; for any fixed $(x, y) \in O^2$ the functions $f(t, x, y)$, $f_x(t, x, y)$, $f_y(t, x, y)$ are measurable on I; for any compact set $K \subset O$ there exists a function $m_K(t) \in L(I, [0, \infty))$ such that

$$|f(t, x, y)| + |f_x(t, x, y)| + |f_y(t, x, y)| \leq m_K(t)$$

for all $(x, y) \in K^2$ and for almost all $t \in I$. Further, let D be the set of continuous differentiable scalar functions (delay functions) $r(t)$, $t \in I$, satisfying the conditions: $\tau(t) < t$, $\dot{\tau}(t) > 0$ with $\widehat{\tau} := \inf\{\tau(a) : \tau(t) \in D\}$ is finite. By Φ we denote the set of continuously differentiable initial functions $\varphi(t) \in M$, $t \in [\widehat{\tau}, b]$.

Let $a < t_{01} < t_{02} < t_{11} < t_{12} < b$ be given numbers and let the functions $q^i(t_0, t_1, x)$, $i = 1, \dots, l$ be continuously differentiable with respect to all arguments $t_0, t_1 \in I$ and $x \in O$.

The collection of initial moment $t_0 \in [t_{01}, t_{02}]$, finally moment $t_1 \in [t_{11}, t_{12}]$, delay function $\tau(t) \in D$ and initial function $\varphi(t) \in \Phi$ is said to be initial data and will be denoted by $w = (t_0, t_1, \tau(t), \varphi(t))$.

To each initial data $w = (t_0, t_1, \tau(t), \varphi(t)) \in W = [t_0, t_1] \times [t_{11}, t_{12}] \times D \times \Phi$ we assign the quasi-linear neutral functional differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t))\big), \ \ t \in [t_0, t_1] \tag{1}$$

with the initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0]. \tag{2}$$

Here $A(t)$ is a given continuous matrix function with dimension $n \times n$ and $\sigma(t) \in D$ is a fixed delay function. The condition (2) is said to be continuous initial condition since always $x(t_0) = \varphi(t_0)$.

Definition 1. Let $w = (t_0, t_1, \tau(t), \varphi(t)) \in W$. A function $x(t) = x(t; w) \in O$, $t \in [\widehat{\tau}, t_1]$, is called the solution of equation (1) with the continuous initial condition (2) or the solution corresponding to the element w and defined on the interval $[\dot{\tau}, t_1]$, if $x(t)$ satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Definition 2. An initial data $w = (t_0, t_1, \tau(t), \varphi(t)) \in W$ is said to be admissible if the corresponding solution $x(t)$ is defined on the interval $[\widehat{\tau}, t_1]$ and the following conditions hold

$$q^i(t_0, t_1, x(t_1)) = 0, \ \ i = 1, \dots, l.$$

The set of admissible initial data will be denoted by W_0.

Definition 3. An initial data $w_0 = (t_{00}, t_{10}, \tau_0(t), \varphi_0(t)) \in W_0$ is said to be optimal if for any $w = (t_0, t_1, \tau(t), \varphi(t)) \in W_0$ we have

$$q^0\big(t_{00}, t_{10}, x_0(t_{10})\big) \le q^0\big(t_0, t_1, x(t_1)\big),$$

where $x_0(t) = x(t; w_0)$, $x(t) = x(t; w)$.

The initial data optimization problem consists in finding an optimal initial data w_0.

Theorem. *Let $w_0 \in W_0$ be an optimal initial data and $t_{00} \in [t_{01}, t_{02}), t_{10} \in (t_{11}, t_{12}]$. Moreover, the function $f(t, x, y), (t, x, y) \in I \times O^2$ is bounded and there exist the finite limits $\dot{x}_0(\sigma(t_{10})-)$,*

$$\lim_{z \to z_0} f(z) = f^+, \;\; z = (t, x, y) \in [t_{00}, t_{10}) \times O^2,$$
$$\lim_{z \to z_1} f(z) = f^-, \;\; z \in (t_{00}, t_{10}] \times O^2,$$

where $z_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00})))$, $z_1 = (t_{10}, x_0(t_{10}), x_0(\tau_0(t_{10})))$. Then there exist a vector $\pi = (\pi_0, \ldots, \pi_l) \ne 0$, $\pi_0 \le 0$ and the solution $(\chi(t), \psi(t))$ of the system

$$\begin{cases} \dot{\chi}(t) = -\psi(t) f_x[t] - \psi(\gamma_0(t)) f_y[\gamma_0(t)] \dot{\gamma}_0(t), & \\ \psi(t) = \chi(t) + \psi(\rho(t)) A(\rho(t)) \dot{\rho}(t), & t \in [t_{00}, t_{10}], \\ \chi(t) = \psi(t) = 0, & t > t_{10} \end{cases} \tag{3}$$

such that the conditions listed below hold:

- *the condition for $\chi(t)$ and $\psi(t)$*

$$\chi(t_{10}) = \psi(t_{10}) = \pi Q_{0x},$$

where $Q = (q^0, \ldots, q^l)^T$, $Q_{0x} = Q_x(t_{00}, t_{10}, x_0(t_{10}))$;

- *the condition for the optimal initial function $\varphi_0(t)$*

$$\chi(t_{00})\varphi_0(t_{00}) + \int_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t)) f_y[\gamma_0(t)] \dot{\gamma}_0(t) \varphi_0(t)\, dt + \int_{\sigma(t_{00})}^{t_{00}} \psi(\rho(t)) A(\rho(t)) \dot{\rho}(t) \dot{\varphi}_0(t)\, dt =$$
$$= \max_{\varphi(t) \in \Phi} \Big[\chi(t_{00})\varphi(t_{00}) + \int_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t)) f_y[\gamma_0(t)] \dot{\gamma}_0(t) \varphi(t)\, dt + \int_{\sigma(t_{00})}^{t_{00}} \psi(\rho(t)) A(\rho(t)) \dot{\rho}(t) \dot{\varphi}(t)\, dt \Big];$$

- *the condition for the optimal delay function $\tau_0(t)$*

$$\int_{t_{00}}^{t_{10}} \psi(t) f_y[t] \dot{x}_0(t) \tau_0(t)\, dt = \min_{\tau(t) \in D} \int_{t_{00}}^{t_{10}} \psi(t) f_y[t] \dot{x}_0(t) \tau(t)\, dt;$$

- *the condition for the optimal initial moment t_{00}*

$$\pi Q_{0t_0} + \psi(t_{00}) \Big[\dot{\varphi}_0(t_{10}) - A(t_{00}) \dot{\varphi}_0(t_{00}) - f^+ \Big] \le 0; \tag{4}$$

- *the condition for the optimal final moment t_{10}*

$$\pi Q_{0t_1} + \psi(t_{10}) \Big[A(t_{10}) \dot{x}_0(\sigma(t_{10})-) + f^- \Big] \ge 0. \tag{5}$$

Here $\gamma_0(t)$ is the inverse function of $\tau_0(t)$ and $\rho(t)$ is the inverse function of $\sigma(t)$;

$$f_x[t] = f_x(t, x_0(t), x_0(\tau_0(t))).$$

Some comments

The essential innovation in this work is necessary optimality condition for delay function. The above given theorem is proved by a scheme described in [1]. Let $f(t,x,y)$ be continuous at the points z_0 and z_1, and let $\dot{x}_0(t)$ be continuous at the point $\sigma(t_{10})$, then instead of inequalities (4) and (5) we have equalities

$$\pi Q_{0t_0} + \psi(t_{00})\Big[\dot{\varphi}_0(t_{10}) - A(t_{00})\dot{\varphi}_0(t_{00}) - f(z_0)\Big] = 0$$

and

$$\pi Q_{0t_1} + \psi(t_{10})\Big[A(t_{10})\dot{x}_0(\sigma(t_{10})) + f(z_1)\Big] = 0.$$

The function $\psi(t)$, generally, is discontinuous at points $\sigma(t_{10}), \sigma(\sigma(t_{10})), \dots$. (see system (3)). The initial data optimization problem for linear neutral functional differential equation with constant delays and the discontinuous initial condition is considered in [2].

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References

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[2] T. Tadumadze, Optimization of initial data for linear neutral functional-differential equations with the discontinuous initial condition. *Azerb. J. Math.* **2** (2012), No. 2, 84–93.