Invariant Sets of Ito Stochastic Systems

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We study invariant sets of Ito stochastic systems

$$dx = a(t,x)dt + \sum_{r=1}^{k} b_r(t,x)dW_r(t), \tag{1}$$

where $t \geq 0$, $x \in \mathbb{R}^n$, $a(t,x)$, $b_r(t,x)$ are in $\mathbb{R}^n$, and $W_1, \ldots, W_r$ are jointly independent scalar Wiener processes defined on a complete probability space (Ω, F, P).

We assume that functions $a(t,x)$ and $b_r(t,x)$ are Borel on the set of variables and Lipschitz in x for $\{t \geq 0\} \times \mathbb{R}^n$ and $a(t,0)$, $b_r(t,0)$ are bounded. It is well known that those conditions assure an existence and uniqueness of a solution of the Cauchy Problem for $t \geq 0$.

Let S be a Borel set in $\{t \geq 0\} \times \mathbb{R}^n$ and $S_t = \{x : (t,x) \in S\}$. Let $S_t \neq \varnothing$ for $t \geq 0$.

Definition 1. The set S is a positive invariant set for the system (1) for $t \geq 0$ if the equality

$$P\Big\{\big(t, x(t,t_0,x_0)\big) \in S, \ \forall\, t \geq t_0\Big\} = 1 \tag{2}$$

holds under the condition $(t_0, x_0(\omega)) \in S$ with $P1$, where $x(t,t_0,x_0)$ is a solution of the system (1) with an initial condition $x(t_0,t_0,x_0) = x_0$, $t_0 \geq 0$.

In other words, if a solution starts in an invariant set, then it remains in the same set.

Remark 1. Note that the set S from the definition (1) is nonrandom (deterministic). Thus we want to obtain the conditions ensuring that the random process "settles" on a deterministic set.

Definition 2. An invariant set S is stochastic stable if for all $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that for $\rho(x_0, S_{t_0}) < \delta$ the next inequality holds

$$P\Big\{\sup_{t \geq t_0} \rho\big(x(t,t_0,x_0), S_t\big) > \varepsilon_1\Big\} < \varepsilon_2. \tag{3}$$

Here is a distance from a point to a set $\rho(x, S_t) = \inf_{y \in S_t} \|x - y\|$.

Let D be a bounded domain in $\mathbb{R}^n$, and a nonnegative Liapunov function $V(t,x)$ be defined for $\{t \geq 0\} \times \overline{D}$ and continuously differentiable twice in x and once in t.

Let N be a set of zeros of $V(t,x)$ in $\{t \geq 0\} \times D$ and $N_t = \{x \in D : V(t,x) = 0\}$. Assume that $N_t \neq \varnothing$ for $t \geq 0$ and let the projection of set N on $\mathbb{R}^n$ be closed in D.

We want to find conditions for the set $N = \{(t,x):\ V(t,x) = 0\}$ to be an invariant set for the system (1). Consider the generating operator for the system (1)

$$LV = \frac{\partial V}{\partial t} + \sum \big(\nabla V, a(t,x)\big) + \frac{1}{2}\sum_{r=1}^{k} \big(\nabla, b_r(t,x)\big)^2 V, \tag{4}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ and $(\,\cdot\,,\,\cdot\,)$ is a scalar product.

Theorem 1. *If the inequality $LV(t,x) \le 0$ holds in domain $\{t \ge 0\} \times D$, then the set*

$$V(t,x) = 0,\ \ t \ge 0,\ \ x \in D \tag{5}$$

is positively invariant for (1)*. If, in addition,*

$$\inf_{t\ge 0,\ x\in D,\ \rho(N_t,x)>\delta} V(t,x) = V_\delta > 0$$

for $\delta > 0$, then the set (5) *is stochastically stable.*

Let the system (1) have positively-invariant set S which is a part of bigger invariant set N, $S \subset N$, and on set N system (1) degenerate into deterministic one.

Definition 3. A set S is stable on N for $t \ge t_0$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_0 \in N$ with $\rho(x_0, S) < \delta$ the next inequality holds

$$\rho(x(t,t_0,x_0), S) < \varepsilon \ \text{ for } \ t \ge t_0. \tag{6}$$

Theorem 2. *Let a positively-invariant set $N \subset D \subset \mathbb{R}^n$ of system* (1) *include a closed positively-invariant set S ($S \subset N$) which is asymptotically-stable on N.*

Then, if the set N is of the form $V(x) = 0$, $x \in D$, $V(x)$ is nonnegative-defined twice continuously-differentiable in $\mathbb{R}^n$ function and

$$LV \le -c_1 V, \tag{7}$$

$$V_r = \inf_{|x|>r} V(x) \to \infty,\ \ r \to \infty, \tag{8}$$

$$|\sigma_r(t,x)|^2 \le c_2 V(x), \tag{9}$$

where $c_1 > 0$, $c_2 > 0$ are constants, then the set S is uniformly stochastically stable for system (1)*. That is for any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exists $\delta = \delta(\varepsilon_1, \varepsilon_2)$ such that for $\rho(x_0, S) < \delta$ the next inequality holds*

$$P\Big\{\sup_{t\ge t_0} \rho\big(x(t,t_0,x_0), S\big) > \varepsilon_1\Big\} < \varepsilon_2. \tag{10}$$

Remark 2. The condition (9) means that the system (1) degenerates into deterministic one on N.

Now we come to an analogue of the Pliss reduction principle.

Let for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $t \ge 0$ we have the Ito system

$$\begin{cases} dx = X(t,y)dt, \\ dy = A(t)ydt + \sigma(t,x,y)dW(t), \end{cases} \tag{11}$$

where X is n-dimensional vector, $A(t)$ is $m\times m$-dimensional matrix, σ is $m\times r$-dimensional matrix, $W(t)$ is r-dimensional Wiener process.

Functions X and σ are Lipschitz over x and y with constants L_1 and L_2, respectively.

Let the fundamental matrix $\Phi(t,s)$ of the system

$$\frac{dy}{dt} = A(t)y \tag{12}$$

satisfy the condition

$$\|\Phi(t,s)\| \leq K \exp\big\{ -\rho(t-s)\big\} \tag{13}$$

for $t \geq s$, $K > 0$, $\rho > 0$.

Let $X(0,0) = 0$ and $\sigma(t,x,0) \equiv 0$. Consequently, $(0,0)$ is a solution of the system (11) and the set $y = 0$ is an invariant set for the system (11). Also on the set $y = 0$ the original stochastic system degenerates into deterministic one

$$dx = X(x,0)dt. \tag{14}$$

We study stability of a trivial solution of the stochastic system (11) using a fact that on the invariant set $y = 0$ the trivial solution is stable as a solution of the deterministic system (14).

Theorem 3. *Let a trivial solution of the system* (14) *be asymptotically stable and* $L_2 < \frac{(2\rho)^{\frac{1}{2}}}{K}$. *Then a trivial solution of the system* (11) *is stochastically stable.*