Differential and Fractional Boundary Value Problems with Strong Time Singularities

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Let $T \in (0, \infty)$ and let $\{\alpha_n\} \subset (0,1)$ be such that $\lim\limits_{n\to\infty} \alpha_n = 1$. We investigate the sequence of singular fractional boundary value problems

$$\frac{\mathrm{d}}{\mathrm{d}t}\, {}^c\!D^{\alpha_n} u(t) = \rho p(t) {}^c\!D^{\alpha_n} u(t) + p(t)a(t)f(t,u(t)), \tag{1}$$

$$u(0) = {}^c\!D^{\alpha_n} u(t)\big|_{t=T}, \quad {}^c\!D^{\alpha_n} u(t)\big|_{t=0} = \frac{a(0)f(0,u(0))}{|\rho|}\,, \tag{2}$$

where $\rho \in (-\infty, 0)$, $f \in C([0,T] \times \mathbb{R})$ and the functions p, a satisfy the condition

(H_1) $p \in C(0,T]$, $a \in C[0,T]$, $p > 0$, $a > 0$ on $(0,T]$ and $\int\limits_0^T p(t)\,\mathrm{d}t = \infty$.

Here, ${}^c\!D$ is the Caputo fractional derivative.

We say that a function $u : [0,T] \to \mathbb{R}$ is *a solution of problem* (1), (2) if ${}^c\!D^{\alpha_n} u \in C[0,T] \cap C^1(0,T]$, u satisfies the boundary conditions (2), and (1) is satisfied for $t \in (0,T]$.

The Caputo fractional derivative ${}^c\!D^\gamma x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x : [0,T] \to \mathbb{R}$ is given as [1, 2]

$${}^c\!D^\gamma x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int\limits_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \Big(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!}\, s^k \Big)\,\mathrm{d}s,$$

where $n = [\gamma] + 1$ and $[\gamma]$ means the integral part of the fractional number γ.

Hence equation (1) can be written in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int\limits_0^t \frac{(t-s)^{-\alpha_n}}{\Gamma(1-\alpha_n)}\,(u(s)-u(0))\,\mathrm{d}s = \rho p(t)\, \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_0^t \frac{(t-s)^{-\alpha_n}}{\Gamma(1-\alpha_n)}\,(u(s)-u(0))\,\mathrm{d}s + p(t)a(t)f(t,u(t)).$$

The special case of (1) is the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, {}^c\!D^{\alpha_n} u(t) = \frac{\rho}{t^\gamma}\, {}^c\!D^{\alpha_n} u(t) + \frac{a(t)f(t,u(t))}{t^\mu}\,,$$

where $\gamma \in [1,\infty)$ and $\mu \in [0,\gamma]$.

Along with problem (1), (2), we discuss the singular differential boundary value problem

$$u''(t) = \rho p(t)u'(t) + p(t)a(t)f(t,u(t)), \tag{3}$$

$$u(0) = u'(T), \quad u'(0) = \frac{a(0)f(0,u(0))}{|\rho|}\,. \tag{4}$$

A function $u \in C^1[0,T] \cap C^2(0,T]$ is called *a solution of problem* (3), (4) if u satisfies (4), and (3) holds for $t \in (0,T]$.

The following result is proved by the Rothe fixed point theorem [3] and gives the existence result for problem (1), (2).

Theorem 1. *Let* (H_1) *hold and let*

(H_2) $f \in C([0,T] \times \mathbb{R})$ *and there exist a positive constant* S *such that for* $t \in [0,T]$ *and* $|x| \le S$, *the estimate*

$$a(t)|f(t,x)| \le \Big(\frac{\max\{1,T\}}{\Delta} + 1\Big)^{-1} |\rho| S$$

is fulfilled, where $\Delta = \min\{\Gamma(\tau) : \ 1 \le \tau \le 2\}$ $(\doteq 0.88)$.

Then for each $n \in \mathbb{N}$ *problem* (1), (2) *has at least one solution* u_n *and*

$$\|u_n\| \le S \quad \text{for } n \in \mathbb{N}.$$

Remark. If f satisfies condition

(H_3) $f \in C([0,T] \times \mathbb{R})$ and for $(t,x) \in [0,T] \times \mathbb{R}$ the estimate

$$a(t)|f(t,x)| \le w(|x|)$$

holds, where $w \in C[0,\infty)$, w is nondecreasing and $\lim\limits_{v\to\infty} w(v)/v = 0$,

then f satisfies condition (H_2).

The relation between solutions of problems (1), (2) and (3), (4) is stated in the following theorem.

Theorem 2. *Let* (H_1) *and* (H_2) *hold. Let* u_n *be a solution of* (1), (2). *Then there exist a subsequence* $\{u_{\ell_n}\}$ *of* $\{u_n\}$ *and a solution* u *of* (3), (4) *such that*

$$\lim_{n\to\infty} \|u_{\ell_n} - u\| = 0, \quad \lim_{n\to\infty} \|{}^c\!D^{\alpha_{\ell_n}} u_{\ell_n} - u'\| = 0.$$

The following results deals with the uniqueness of solutions to problems (1), (2) and (3), (4).

Theorem 3. *Let* (H_1) *hold. Let* $f \in C([0,T] \times \mathbb{R})$ *and*

$$|f(t,x) - f(t,y)| \le K|x-y| \quad \text{for } t \in [0,T] \text{ and } x,y \in \mathbb{R},$$

where

$$K < \frac{|\rho|}{(T+1)\|a\|}\,.$$

Then for all sufficiently large n *problem* (1), (2) *has a unique solution* u_n *and*

$$\lim_{n\to\infty} \|u_n - u\| = 0, \quad \lim_{n\to\infty} \|{}^c\!D^{\alpha_n} u_n - u'\| = 0,$$

where u *is the unique solution of* (3), (4).

References

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