On Existence of a Special Kind's Integral Manifold of the Nonlinear Differential System, Containing Slowly Varying Parameters

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Let $G(\varepsilon_0) = \{t, \varepsilon :\ t \in \mathbf{R},\ \varepsilon \in [0,\ \varepsilon_0],\ \varepsilon_0 \in \mathbf{R}^+\}$.

Definition 1. We say that a function $f(t,\varepsilon)$, in general a complex-valued, belongs to the class $S_m(\varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$, if

(1) $f : G(\varepsilon_0) \to \mathbf{C}$;

(2) $f(t,\varepsilon) \in C^m(G(\varepsilon_0))$ with respect to t;

(3) $d^k f(t,\varepsilon)/dt^k = \varepsilon^k f_k^*(t,\varepsilon)$ $(0 \le k \le m)$,

$$\|f\|_{S_m(\varepsilon_0)} \overset{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |f_k^*(t,\varepsilon)| < +\infty.$$

Definition 2. We say that a function $f(t,\varepsilon,\theta)$ belongs to the class $F_m^\theta(\varepsilon_0,\alpha)$ $(m \in \mathbf{N} \cup \{0\}$, $\alpha \in (0,+\infty))$ if

(1) $t, \varepsilon \in G(\varepsilon_0)$, $\theta \in \mathbf{R}$;

(2) $f : G(\varepsilon_0) \times \mathbf{R} \to \mathbf{R}$;

(3)

$$f(t,\varepsilon,\theta) = \sum_{n=-\infty}^{\infty} f_n(t,\varepsilon)\exp(in\theta),$$

and

(a) $f_n(t,\varepsilon) \in S_m(\varepsilon_0)$, $f_{-n}(t,\varepsilon) \equiv \overline{f_n(t,\varepsilon)}$;

(b) $\exists K \in (0,+\infty)$: $\|f_n\|_{S_m(\varepsilon_0)} \le K\exp(-|n|\alpha)$, $n \in \mathbf{Z}$;

(c)

$$\|f\|_{F_m^\theta(\varepsilon_0,\alpha)} \overset{def}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S_m(\varepsilon_0)} < \frac{K(1+e^{-\alpha})}{1-e^{-\alpha}}\,.$$

So the function $f(t,\varepsilon,\theta)$ and its partial derivatives with respect to t up to m-th order inclusive are analytic with respect to $\theta \in \mathbf{R}$.

Definition 3. We say that a function $f(t,\varepsilon,x)$ belongs to the class $S_m^x(\varepsilon_0,x_0,d)$, if

(1) $t, \varepsilon \in G(\varepsilon_0)$, $x \in \mathbf{R}$;

(2) $f : G(\varepsilon_0) \times \mathbf{R} \to \mathbf{R}$;

(3)

$$f(t,\varepsilon,x)=\sum_{l=0}^{\infty} f_l(t,\varepsilon)(x-x_0)^l,$$

and

(a) $f_l : G(\varepsilon_0)\to \mathbf{R}$;

(b) $f_l(t,\varepsilon)\in S_m(\varepsilon_0)$;

(c) the series $\sum\limits_{l=0}^{\infty}\|f_l\|_{S_m(\varepsilon_0)}(x-x_0)^l$ is convergent if $|x-x_0|<d$.

Thus the function $f(t,\varepsilon,x)$ is real, analytic with respect to x, if $|x-x_0|<d$ together with its partial derivatives up to m-th order inclusive. Moreover, $\forall\, x\in(x_0-d,x_0+d) : f(t,\varepsilon,x)\in S_m(\varepsilon_0)$.

Definition 4. We say that a function $f(t,\varepsilon,\theta,x)$ belongs to the class $F_m^{\theta,x}(\varepsilon_0,\alpha,x_0,d)$ ($m\in \mathbf{N}\cup\{0\}$, $\alpha\in(0,+\infty)$) if

(1) $t,\varepsilon\in G(\varepsilon_0)$, $\theta\in\mathbf{R}$, $x\in\mathbf{R}$;

(2) $f : G(\varepsilon_0)\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$;

(3)

$$f(t,\varepsilon,\theta,x)=\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty} f_{n,l}(t,\varepsilon)e^{in\theta}(x-x_0)^l,$$

and

(a) $f_{n,l}(t,\varepsilon)\in S_m(\varepsilon_0)$, $f_{-n,l}(t,\varepsilon)\equiv\overline{f_{n,l}(t,\varepsilon)}$,

(b) $\exists\, K\in(0,+\infty) : \ \forall\, n\in\mathbf{Z}$, $\forall\,\rho\in(0,d)$:

$$\|f_{n,l}(t,\varepsilon)\|_{S_m(\varepsilon)}\le\frac{Ke^{-|n|\alpha}}{\rho^l}\,.$$

We denote

$$X_0(t,\varepsilon,x)=\frac{1}{2\pi}\int\limits_0^{2\pi} X(t,\varepsilon,\theta,x)\,d\theta.$$

Consider the following system of differential equations

$$\begin{aligned}\frac{dx}{dt}&=\mu X(t,\varepsilon,\theta,x)+\varepsilon a(t,\varepsilon,\theta,x),\\ \frac{d\theta}{dt}&=\omega(t,\varepsilon)+\mu\Theta(t,\varepsilon,\theta,x)+\varepsilon b(t,\varepsilon,\theta,x),\end{aligned}\tag{1}$$

where $t,\varepsilon\in G(\varepsilon_0)$, $\theta,x\in\mathbf{R}$; $X,\Theta\in F_m^{\theta,x}(\varepsilon_0,\alpha,x_0,d)$, $a,b\in F_{m-1}^{\theta,x}(\varepsilon_0,\alpha,x_0,d)$, $\omega\in S_m(\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\omega=\omega_0>0$, $\mu\in(0,\mu_0)$.

We study the question of the existence of the integral manifold $x=w(t,\varepsilon,\theta,\mu)\in F_k^{\theta}(\varepsilon_1,\alpha_1)$ ($k<m-1$, $\varepsilon_1<\varepsilon_0$, $\alpha_1<\alpha$) of the system (1).

Let us assume that the following conditions hold.

(A) There is a real function $x_0(t,\varepsilon)$ such that

(1) $X_0(t,\varepsilon,x_0(t,\varepsilon))\equiv 0$;

(2)

$$\inf_{G(\varepsilon_0)} \left| \frac{\partial X_0(t,\varepsilon,x_0(t,\varepsilon))}{\partial x} \right| = \gamma > 0; \tag{2}$$

(3) in system (1) a function $x_0(t,\varepsilon)$ is taken as a point x_0 and is taken as d – sufficiently small positive number in the d-neighborhood of the point x_0 is no other roots of the equation $X_0(t,\varepsilon,x) = 0$, than x_0. Owing to the condition (2) the number d are exists.

(B) Parameters μ and ε are related by inequalities

$$\mu^{r-2} \le \varepsilon^{m_1-1}, \tag{3}$$

where $r, m_1 \in \mathbf{N}$, $r > 2m_1$, $m > 2m_1$, $m_1 \ge 1$,

$$\mu + \frac{\varepsilon}{\mu^2} < \delta, \tag{4}$$

where $\delta \in (0,+\infty)$.

Theorem. *Suppose that the system* (1) *satisfies conditions* (A), (B). *Then* $\exists\, \delta_0 \in (0,+\infty)$ *such that* $\forall\, \delta \in (0,\delta_0)$ (δ *– value in condition* (B)) *the system* (1) *has the integral manifold*

$$x = w(t,\varepsilon,\theta,\mu) \in F^{\theta}_{m_1-1}(\varepsilon_1^*, \alpha^*),$$

where $\varepsilon_1^* \in (0,\varepsilon_0)$, $\alpha^* \in (0,\alpha)$, *and on this manifold the system* (1) *is reduced to the equation*

$$\frac{d\theta}{dt} = \omega(t,\varepsilon) + \mu\Theta(t,\varepsilon,\theta,w(t,\varepsilon,\theta,\mu)) + \varepsilon b(t,\varepsilon,\theta,w(t,\varepsilon,\theta,\mu)).$$