Construction of Periodic Solutions and Interval Halving Procedure

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We are interested in a constructive numerical-analytic method of investigation of the periodic boundary value problem

$$u'(t) = f(t, u(t)), \quad t \in [a, b], \tag{1}$$

$$u(b) = u(a), \tag{2}$$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function Lipschitzian with respect to the second variable:

$$\big|f(t, z_1) - f(t, z_2)\big| \leq K|z_1 - z_2| \tag{3}$$

for all z_1, z_2 from a certain bounded set D. The inequality and the absolute value sign in (3) are understood componentwise. The numerical-analytic scheme based on the successive approximations (see, e.g., the references in [1])

$$u_m(t, z) := z + \int\limits_a^t f\big(s, u_{m-1}(s, z)\big)\, ds - \frac{t-a}{b-a} \int\limits_a^b f\big(s, u_{m-1}(s, z)\big)\, ds, \tag{4}$$

where $u_0(t, z) := z$, $t \in [a, b]$, $m = 1, 2, \dots$, $z \in D$, allows one to both study the solvability of the problem and practically construct approximations to its solution. The idea of use of iterations (4) to study problem (1), (2) is based on the fact that, in the case of solvability, the free parameter z involved in (4) can always be chosen so that the limit of sequence (4) is a solution of the problem. The parameter z plays the role of an unknown initial value of the solution.

In order to guarantee the applicability of this approach, the previous works contained the assumptions that the eigenvalues of the Lipschitz matrix are sufficiently small and the domain D where (3) holds is, in a sense, wide enough. In particular, the method based on (4) is known to be convergent provided that

$$r(K) < \frac{10}{3(b-a)}\,. \tag{5}$$

We have shown recently in [2] that, at some computational expence, the scheme of the method can be modified so that the convergence condition becomes twice as weak as the original one. More precisely, the mentioned modified version converges provided that

$$r(K) < \frac{20}{3(b-a)}\,, \tag{6}$$

which is a considerably weaker assumption. The idea is to construct the iterations from suitable sequences defined on the half-intervals.

Let Ω be a closed convex subset of D, in which one looks for initial values of periodic solutions. We assume that f satisfies the Lipshitz (3) condition on D. Note that the main role is now played by Ω, and not D. Let ξ and η be arbitrary vectors from Ω. By analogy to [2], put

$$x_0(t,\xi,\eta) := \Big(1 - \frac{2(t-a)}{b-a}\Big)\xi + \frac{2(t-a)}{b-a}\,\eta, \quad t \in [a,(a+b)/2],$$
$$y_0(t,\xi,\eta) := \Big(1 - \frac{2(t-a-b)}{b-a}\Big)\xi + \frac{2t-a-b}{b-a}\,\xi, \quad t \in [(a+b)/2,b],$$

define the recurrence sequences of functions $x_m : [a,(a+b)/2] \times \Omega^2 \to \mathbb{R}^n$ and $y_m : [(a+b)/2,b] \times \Omega^2 \to \mathbb{R}^n$, $m = 0,1,\dots$, according to the formulas

$$x_m(t,\xi,\eta) := x_0(t,\xi,\eta) + \int_a^t f\big(s, x_{m-1}(s,\xi,\eta)\big)\,ds - \frac{2(t-a)}{b-a}\int_a^{\frac{a+b}{2}} f\big(s, x_{m-1}(s,\xi,\eta)\big)\,ds, \quad t \in [a,(a+b)/2], \tag{7}$$

$$y_m(t,\xi,\eta) := y_0(t,\xi,\eta) + \int_{\frac{a+b}{2}}^t f\big(s, y_{m-1}(s,\xi,\eta)\big)\,ds - \frac{2t-a-b}{b-a}\int_{\frac{a+b}{2}}^{b} f\big(s, y_{m-1}(s,\xi,\eta)\big)\,ds, \quad t \in [(a+b)/2,b], \tag{8}$$

where $m \geq 0$. Note the presence of the two parameter vectors, ξ and η, in (7), (8), in contrast to one appearing in (4).

Let
$$\delta_\Omega(f) := \max\Big\{\delta_{[a,(a+b)/2],\Omega}(f),\ \delta_{[(a+b)/2,b],\Omega}(f)\Big\}, \tag{9}$$
where $\delta_{J,\Omega}(f) := \max\limits_{(t,z)\in J\times\Omega} f(t,z) - \min\limits_{(t,z)\in J\times\Omega} f(t,z)$ for any closed interval $J \subseteq [a,b]$. Given a non-negative vector ϱ, put $\Omega_\varrho := \bigcup\limits_{\xi\in\Omega} B(\xi,\varrho)$, where $B(\xi,\varrho) := \{z \in \mathbb{R}^n : \ |\xi - z| \leq \varrho\}$.

Theorem 1. *If the spectral radius of K satisfies* (6) *and*

$$\exists\, \varrho : \quad \Omega_\varrho \subset D \ \text{ and } \ \varrho \geq \frac{b-a}{8}\,\delta_{\Omega_\varrho}(f) \tag{10}$$

then, for all fixed $(\xi,\eta) \in \Omega^2$, the sequence $\{x_m(\,\cdot\,,\xi,\eta) : m \geq 0\}$ (resp., $\{y_m(\cdot,\xi,\eta) : m \geq 0\}$) converges to a limit function $x_\infty(\,\cdot\,,\xi,\eta)$ (resp., $y_\infty(\,\cdot\,,\xi,\eta)$) uniformly in $t \in [a,(a+b)/2]$ (resp., $t \in [(a+b)/2,b]$).

Condition is twice as weak as the original inequality (5). Furthermore, comparing condition (10) with similar assumptions from earlier works, we find that (10) is easier to verify because in order to do so one has only to find the value $\delta_{\Omega_\varrho}(f)$, which is computed directly by estimating f. It is also clear from (9) that the value $\delta_{\Omega_\varrho}(f)$ does not change when D grows.

Theorem 2. *If* (6) *and* (10) *hold, then the function $u_\infty(\,\cdot\,,\xi,\eta) : [a,b] \to \mathbb{R}^n$ defined by the formula*

$$u_\infty(t,\xi,\eta) := \begin{cases} x_\infty(t,\xi,\eta) & \text{if } t \in [a,(a+b)/2], \\ y_\infty(t,\xi,\eta) & \text{if } t \in [(a+b)/2,b], \end{cases}$$

for $(\xi,\eta)\in\Omega^2$ *is a solution of problem* (1), (2) *if and only if*

$$\Xi(\xi,\eta)=0,\quad H(\xi,\eta)=0,$$

where

$$\Xi(\xi,\eta):=\eta-\xi-\int_a^{\frac{a+b}{2}} f\big(\tau,x_\infty(\tau,\xi,\eta)\big)\,d\tau,$$

$$H(\xi,\eta):=\xi-\eta-\int_{\frac{a+b}{2}}^{b} f\big(\tau,y_\infty(\tau,\xi,\eta)\big)\,d\tau.$$

Moreover, for every solution $u(\cdot)$ *of problem* (1), (2) *with* $(u(a),u(\frac{a+b}{2}))\in\Omega^2$*, there exists a pair* (ξ_0,η_0) *in* Ω^2 *such that* $u(\cdot)=u_\infty(\,\cdot\,,\xi_0,\eta_0)$.

Theorems 1 and 2 suggest a scheme of investigation of the periodic boundary value problem (1), (2), which can be realised on practice by using certain approximate determining functions considered for a finite number of step and, thus, computable explicitly. These approximate determining functions also allow one to obtain constructive conditions guaranteeing the solvability of problem (1), (2) (see [2]).

Multiple interval divisions can be carried out, which, at the price of increase of the number of determining equations to be solved numerically, proportionally diminishes the constants in the conditions. For example, with $[a,b]$ divided into 4 parts at once, we replace (6) by the weaker condition

$$r(K)<\frac{40}{3(b-a)}\,. \tag{11}$$

The condition on the neighbourhood of Ω also becomes proportionally weaker: instead of (10), one arrives at the assumption that

$$\exists\,\varrho:\quad \Omega_\varrho\subset D\ \text{ and }\ \varrho\geq\frac{b-a}{16}\,\delta_{\Omega_\varrho}(f). \tag{12}$$

Assumption (12) is, clearly, weaker than (10).

References

[1] A. Rontó and M. Rontó, Successive approximation techniques in non-linear boundary value problems for ordinary differential equations. *Handbook of differential equations: ordinary differential equations. Vol. IV*, 441–592, Handb. Differ. Equ., *Elsevier/North-Holland, Amsterdam*, 2008.

[2] A. Rontó, M. Rontó, and N. Shchobak, Constructive analysis of periodic solutions with interval halving. *Bound. Value Probl.* **2013**, 2013:57, 34 pp.