Boundary Value Problems with State-Dependent Impulses

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Impulsive differential equations have attracted lots of interest due to their important applications in many areas such as aircraft control, drug administration, and threshold theory in biology. A particular case of impulsive problems are *problems with impulses at fixed moments.* This occurs when the moments, at which impulses act in state variable, are known. Very different situation arises, when the impulses appear in evolutionary trajectories fulfilling a predetermined relation between state and time variables. This case, which is represented by state-dependent impulses, is discussed here. Studies of real-life problems with state-dependent impulses can be found e.g. in [1], [3]–[5].

In particular, here we investigate the solvability of *boundary value problems with state-dependent impulses.* As the methods used for problems with finitely many impulses acting at fixed points do not apply to problems with state-dependent impulses, only few paper dealing with boundary value problems in the state-dependent case may be found in the literature. Most of them consider periodic problems which can be transformed to fixed point problems of corresponding Poincaré maps. So, in the case of a periodic boundary conditions, difficulties with the construction of a proper function space and a proper operator representation have been cleared, see e.g. [2].

The main cause of difficulties in the investigation of boundary value problems with state-dependent impulses lies in the following fact: the operator, corresponding to the problem with state-dependent impulses which is constructed in a standard way (used for problems with fixed-time impulses), is not continuous. Therefore, in [6]–[9] we provide a new approach which makes possible to find sufficient conditions for solvability of the ordinary differential equation

$$z''(t) = f(t, z(t), z'(t)) \quad \text{for a.e. } t \in [a, b], \tag{1}$$

subject to the impulse conditions

$$z(\tau_i+) - z(\tau_i) = J_i(\tau_i, z(\tau_i)), \quad z'(\tau_i+) - z'(\tau_i-) = M_i(\tau_i, z(\tau_i)), \tag{2}$$

where the points $\tau_1, \dots, \tau_p$ depend on z through the equations

$$\tau_i = \gamma_i(z(\tau_i)), \quad i = 1, \dots, p, \ \ p \in N.$$

Problem (1), (2) is studied together with the general linear boundary conditions

$$\ell_1(z, z') = c_1, \quad \ell_2(z, z') = c_2. \tag{3}$$

Here f fulfils the Carathéodory conditions on $[a, b] \times R^2$, the impulse functions J_i, M_i, $i = 1, \dots, p$, are continuous on $[a, b] \times R$, $c = (c_1, c_2) \in R^2$, and ℓ_1, ℓ_2 are linear and bounded functionals in the space $G_L([a, b]; R^2)$ of left-continuous regulated vector functions. Consequently, $\ell = (\ell_1, \ell_2)$ is represented by the formula containing the Kurzweil–Stieltjes integral

$$\ell(x) = Kx(a) + \int\limits_a^b V(t) d\,[x(t)], \ \ x = (x_1, x_2) \in G_L([a, b]; R^2),$$

where K is a constant matrix and elements of a function matrix V are functions of bounded variation on $[a, b]$. The barriers γ_i, $i = 1, \dots, p$, which determine the impulse points τ_i, $i = 1, \dots, p$, are ordered and differentiable on some compact real interval.

A function $u : [a, b] \to R$ is a solution of problem (1)–(3) if for each $i \in \{1, \dots, p\}$ there exists a unique $\tau_i \in (a, b)$ such that

$$\tau_i = \gamma_i(u(\tau_i)), \quad a < \tau_1 < \tau_2 < \dots < \tau_p < b,$$

the restrictions $u|_{[a,\tau_1]}$, $u|_{(\tau_1,\tau_2]}$, ..., $u|_{(\tau_p,b]}$ have absolutely continuous first derivatives, u satisfies (1) for a.e. $t \in [a, b]$ and fulfils conditions (2) and (3).

Provided the data functions f, J_i, M_i, $i = 1, \dots, p$ are bounded, transversality conditions which guarantee that each possible solution of equation (1) in a given region crosses each barrier γ_i at a unique impulse point τ_i are presented, and consequently the existence of a solution to problem (1)–(3) is proved. In order to do it, we choose the way which we have developed in our joint papers [6]–[9]. The main idea of our approach lies in the representation of the solution u of problem (1)–(3) by an ordered $(p + 1)$-tuple of functions smooth on $[a, b]$. In particular, we define a Banach space $X = [C^1([a, b]; R)]^{p+1}$, a set $\Omega \subset X$ and an operator $F : \bar{\Omega} \to X$ which is compact. We prove the existence of a fixed point $(u_1, \dots, u_{p+1}) \in \bar{\Omega}$ of the operator F, and then we construct a solution u of problem (1)–(3) by means of this fixed point.

Such existence result can be extended to unbounded functions f, J_i, M_i, $i = 1, \dots, p$ by means of the method of a priori estimates. This can be found for the special case of ℓ in [9].

References

[1] F. Córdova-Lepe, M. Pinto, and E. González-Olivares, A new class of differential equations with impulses at instants dependent on preceding pulses. Applications to management of renewable resources. *Nonlinear Anal. Real World Appl.* **13** (2012), No. 5, 2313–2322.

[2] I. Bajo and E. Liz, Periodic boundary value problem for first order differential equations with impulses at variable times. *J. Math. Anal. Appl.* **204** (1996), No. 1, 65–73.

[3] J. Jiao, Sh. Cai, and L. Chen, Analysis of a stage-structured predatory-prey system with birth pulse and impulsive harvesting at different moments. *Nonlinear Anal. Real World Appl.* **12** (2011), No. 4, 2232–2244.

[4] L. Nie, Zh. Teng, L. Hu, and J. Peng, Qualitative analysis of a modified Leslie-Gower and Holling-type II predator-prey model with state dependent impulsive effects. *Nonlinear Anal. Real World Appl.* **11** (2010), No. 3, 1364–1373.

[5] L. Nie, Zh. Teng, and A. Torres, Dynamic analysis of an SIR epidemic model with state dependent pulse vaccination. *Nonlinear Anal. Real World Appl.* **13** (2012), No. 4, 1621–1629.

[6] I. Rachůnková and J. Tomeček, A new approach to BVPs with state-dependent impulses. *Bound. Value Probl.* **2013**, 2013:22, 13 pp.

[7] I. Rachůnková and J. Tomeček, Second order BVPs with state dependent impulses via lower and upper functions. *Cent. Eur. J. Math.* **12** (2014), No. 1, 128–140.

[8] I. Rachůnková and J. Tomeček, Impulsive system of ODEs with general linear boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2013**, No. 25, 16 pp.

[9] I. Rachůnková and J. Tomeček, Existence principle for BVPs with state-dependent impulses. (submitted).