On a Certain Discontinuous Dynamical System in the Plane

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In this article we investigate the behavior of solutions of a certain discontinuous dynamical system in the plane:

$$\dot{x} = Jx, \quad \langle a, x\rangle \neq 0; \quad \Delta x\big|_{\langle a,x\rangle=0} = Bx. \tag{1}$$

Here $x = col(x_1, x_2)$, $\langle a, x\rangle = a_1x_1 + a_2x_2 = 0$ is a given line in the plane, $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ are constant matrices.

The act of motion of the phase point $(x_1(t), x_2(t))$ is defined by the given system of differential equations $\dot{x} = Jx$, when this point is out of the line $x_2 = kx_1$, $k = -\frac{a_1}{a_2}$, and at the time $t = t^*$, when $x_2(t^*) = kx_1(t^*)$, phase point "instantly" jumps to a point

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^+ = \begin{pmatrix} 1+b_{11} & b_{12} \\ b_{21} & 1+b_{22} \end{pmatrix} \begin{pmatrix} x_1(t^*) \\ x_2(t^*) \end{pmatrix}.$$

Note that the linear homogeneous transformation

$$(E+B): \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1+b_{11} & b_{12} \\ b_{21} & 1+b_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

of the plane (x_1Ox_2) into itself maps the line $x_2 = kx_1$ into the line $x_2 = \mu x_1$, where the angular coefficients k and μ relate to equality:

$$\mu = \frac{k(1+b_{22}) + b_{21}}{1 + b_{11} + kb_{12}}.$$

The phase point will necessarily meet the line $x_2 = \mu x_1$ despite the initial position. So it is enough to investigate the behavior of solutions that begin on this line.

Without loss of generality, we consider only the case when $k > 0$. Other cases are investigated similarly.

Since the solutions of differential systems are explicitly written out, simple cases reduce the investigation of behavior of discontinuous trajectories to the study the Poincare mapping

$H : R \to R$ line into a line:

$$H : x_1 \to e^{\frac{\alpha}{\beta}(\operatorname{arctg}\mu - \operatorname{arctg}k)} \sqrt{\frac{1+\mu^2}{1+k^2}}\,(1 + b_{11} + kb_{12})x_1, \quad \text{if } \mu > k > 0,$$

and

$$H : x_1 \to e^{\frac{\alpha}{\beta}(\pi + \operatorname{arctg}\mu - \operatorname{arctg}k)} \sqrt{\frac{1+\mu^2}{1+k^2}}\,(1 + b_{11} + kb_{12})x_1, \quad \text{if } \mu \le k.$$

Denote by h the relation

$$h = \begin{cases} e^{\frac{\alpha}{\beta}(\operatorname{arctg}\mu - \operatorname{arctg}k)} \sqrt{\frac{1+\mu^2}{1+k^2}}\,(1 + b_{11} + kb_{12})x_1, & \text{if } \mu > k > 0 \\ e^{\frac{\alpha}{\beta}(\pi + \operatorname{arctg}\mu - \operatorname{arctg}k)} \sqrt{\frac{1+\mu^2}{1+k^2}}\,(1 + b_{11} + kb_{12})x_1, & \text{if } \mu \le k \end{cases}.$$

Theorem 1. *If the parameters of the original discontinuous dynamical system* (1) *are such that* $|h| < 1$*, then all its solutions over time tends to zero; if* $|h| > 1$*, then all solutions tends to infinity as* $t \to \infty$*. If* $h = 1$*, then all of the solutions which touch the line* $x_2 = \mu x_1$ *are periodic with one impulsive perturbations per period* (*all trajectories are one-impulsive cycles*)*. If* $h = -1$*, then the system has an one-parameter family of two-impulsive cycles, which were generated by discontinuous periodic solutions with two breaks per period.*

As an example, consider the possibility of undamped oscillations of a linear oscillator with friction

$$\ddot{x} + 2\alpha\dot{x} + \omega^2 x = 0, \quad \alpha > 0, \quad \omega^2 > \alpha^2.$$

We assume that the perturbation of oscillator subjected to impulsive perturbations at the moment, when the instantaneous speed of the phase point is zero, and the value of the pulse action is proportional to the strength of the coefficients γ position of the phase point at this moment, namely,

$$\Delta\dot{x}\big|_{\dot{x}=0} = \gamma x, \quad \delta x\big|_{\dot{x}=0}.$$

Having the replacement

$$y = \frac{1}{\beta}\,(\dot{x} + \alpha x), \quad \beta = \sqrt{\omega^2 - \alpha^2}\,,$$

we obtain a system of the form (1)

$$\begin{cases} \dot{x} = -\alpha x + \beta y \\ \dot{y} = -\beta x - \alpha y, \quad y \neq \dfrac{\alpha}{\beta}\,x \end{cases},$$

$$\Delta y\big|_{y=\frac{\alpha}{\beta}\,x} = \frac{\gamma}{\beta}\,x.$$

Here

$$J = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}, \quad B = \begin{pmatrix} -\beta & -\alpha \\ -\dfrac{\gamma}{\beta} & 0 \end{pmatrix}, \quad \begin{cases} K = \dfrac{\alpha}{\beta} \\ \mu = \dfrac{\alpha + \gamma}{\beta} \end{cases}.$$

Direct calculations show that when $\mu > k$, the system has no one-impulsive and two-impulsive cycles.

If $\mu < k$, namely $\gamma < 0$, one-impulsive cycles in the system do not exist, but there are two-impulsive in case $\gamma = \gamma^*$, where γ^* is a negative root of the equation

$$e^{-2\frac{\alpha}{\beta}}\Big(\pi + \operatorname{arctg}\Big(k + \frac{\gamma}{\beta}\Big) - \operatorname{arctg}(k)\Big) = \frac{1 + k^2}{1 + (\frac{\gamma}{\beta})^2}\,.$$

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