New Results on Preservation of Invariant Tori of Nonlinear Multi-Frequency Systems

Mykola Perestyuk and Petro Feketa

Taras Shevchenko National University of Kyiv, Ukraine
E-mail: pmo@univ.kiev.ua, petro.feketa@gmail.com

One of the important issues of a theory of multi-frequency oscillations is roughness of invariant manifold and its preservation under small perturbations [6]. In numerous papers (e.g. [1, § 10]) a direct Lyapunov method was utilized for the investigations of roughness of invariant toroidal manifold. It was proved that a sufficiently small perturbation of right-hand side of system do not ruin the invariant torus.

Here were have established new conditions for the preservation of the asymptotically stable invariant toroidal manifold that demand the perturbation to be small not on the whole surface of a torus $\mathcal{T}_m$, but only in non-wandering set of dynamical system on torus (see [4] for details).

Consider the perturbed system of differential equations defined in the direct product of a torus $\mathcal{T}_m$ an an Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = (A + B(\varphi))x + f(\varphi), \tag{1}$$

where $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$, $a(\varphi) \in C_{Lip}(\mathcal{T}_m)$, A is a constant matrix, $B(\varphi), f(\varphi) \in C(\mathcal{T}_m)$. Our goal is to establish new sufficient conditions for the existence of invariant torus of system (1) when an unperturbed system

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = Ax + f(\varphi), \tag{2}$$

has an asymptotically stable invariant toroidal manifold. As it is known [1, § 10], system (1) has an invariant torus for an arbitrary function $f(\varphi) \in C(\mathcal{T}_m)$, if a perturbation term $B(\varphi)$ is sufficiently small for every $\varphi \in \mathcal{T}_m$. We are weakening this condition and demand that $\|B(\varphi)\| \leq \delta$ only for $\varphi \in \Omega$, where Ω is a non-wandering set of dynamical system $\frac{d\varphi}{dt} = a(\varphi)$.

Definition 1. A point φ is called non-wandering if there exist a neighbourhood $U(\varphi)$ and a positive constant T such that

$$U(\varphi) \cdot \varphi_t(U(\varphi)) = 0 \quad \text{for} \ \ t \geq T. \tag{3}$$

Denote by W and $\Omega = \mathcal{T}_m - W$ a set of wandering and non-wandering points respectively. From compactness of torus it follows that a set Ω is non-empty and compact.

Theorem 1. *Let in system* (1) *real parts of all eigenvalues of matrix A be negative:* $\operatorname{Re}\lambda_j(A) < 0$, $j = 1, \ldots, n$. *Then there exists $\delta > 0$ such that for an arbitrary function $B(\varphi) \in C(\mathcal{T}_m)$ such that $\|B(\varphi)\| \leq \delta$, $\varphi \in \Omega$ and for an arbitrary function $f(\varphi) \in C(\mathcal{T}_m)$, system* (1) *has an asymptotically stable invariant toroidal manifold.*

From theorem 1 it follows an important corollary that allows to investigate a qualitative behavior of solutions of complex systems that have simple dynamics in non-wandering set Ω.

Corollary 1. *Consider the system*

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \tag{4}$$

where $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$, $a(\varphi) \in C_{Lip}(\mathcal{T}_m)$, $A(\varphi), f(\varphi) \in C(\mathcal{T}_m)$. Let matrix $A(\varphi)$ be constant in non-wandering set Ω and real parts of all eigenvalues of constant matrix be negative. Then for an arbitrary function $f(\varphi) \in C(\mathcal{T}_m)$ system (4) *has an asymptotically stable invariant toroidal manifold.*

Utilizing a classical perturbation theory for multi-frequency systems it is easy to prove a sufficient conditions for the existence of asymptotically stable invariant toroidal manifold of nonlinear system of the form

$$\frac{d\varphi}{dt} = a(\varphi) + a_1(\varphi, x), \quad \frac{dx}{dt} = A(\varphi)x + F(\varphi, x), \tag{5}$$

where $\varphi \in \mathcal{T}_m,\ x \in \overline{J}_h,\ a(\varphi) \in C_{Lip}(\mathcal{T}_m),\ a_1(\varphi, x) \in C_{Lip}(\mathcal{T}_m, \overline{J}_h),\ F(\varphi, x) \in C^{(0,2)}(\mathcal{T}_m, \overline{J}_h)$, $\overline{J}_h = \{x \in \mathbb{R}^n,\ \|x\| \le h,\ h > 0\}$. System (5) may be rewritten in the form

$$\frac{d\varphi}{dt} = a(\varphi) + a_1(\varphi, x), \quad \frac{dx}{dt} = A(\varphi)x + A_1(\varphi, x)x + f(\varphi), \tag{6}$$

where $A_1(\varphi, x) = \int\limits_0^1 \frac{\partial F(\varphi, \tau x)}{\partial(\tau x)}\, d\tau,\ f(\varphi) = F(\varphi, 0)$.

Corollary 2. *Let in system* (5) *matrix* $A(\varphi)$ *be constant in non-wandering set* Ω *and real parts of all eigenvalues of constant matrix be negative:*

$$A(\varphi)|_{\varphi \in \Omega} = \widetilde{A}, \quad \operatorname{Re} \lambda_j(\widetilde{A}) < 0, \ \ j = 1, \dots, n.$$

Then there exist sufficiently small constants η *and* δ *and sufficiently small Lipschitz constants* L_1 *and* L_2 *such that for an arbitrary functions* $a_1(\varphi, x) \in C_{Lip}(\mathcal{T}_m, \overline{J}_h)$, $F(\varphi, x) \in C^{(0,2)}(\mathcal{T}_m, \overline{J}_h)$ *such that*

$$\max_{\varphi \in \mathcal{T}_m,\, x \in \overline{J}_h} \|a_1(\varphi, x)\| \le \eta, \qquad \max_{\varphi \in \mathcal{T}_m,\, x \in \overline{J}_h} \|A_1(\varphi, x)\| \le \delta,$$
$$\big\|a_1(\varphi, x') - a_1(\varphi, x'')\big\| \le L_1 \|x' - x''\|, \quad \big\|A_1(\varphi, x') - A_1(\varphi, x'')\big\| \le L_2 \|x' - x''\|,$$

for any $x', x'' \in \overline{J}_h$, *system* (5) *has an asymptotically stable invariant toroidal manifold.*

Consider a case when function $A(\varphi_t(\varphi))$ is periodic with respect to t for $\varphi \in \Omega$. For example, such a situation appears when a set Ω consists only from a single trajectory that is a cycle.

Corollary 3. *Consider the system*

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \tag{7}$$

where $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$, $a(\varphi) \in C_{Lip}(\mathcal{T}_m)$, $A(\varphi), f(\varphi) \in C(\mathcal{T}_m)$. *Let matrix* $A(\varphi_t(\varphi))$ *be a periodic with respect to* t *for* $\varphi \in \Omega$ *and all the multipliers of linear periodic system* $\frac{dx}{dt} = A(\varphi_t(\varphi))x$, $\varphi \in \Omega$ *lie inside the unit circle. Then for an arbitrary function* $f(\varphi) \in C(\mathcal{T}_m)$ *system* (7) *has an asymptotically stable invariant toroidal manifold.*

Note that one should built a fundamental matrix of periodic system to get the multipliers. In general case it could be very difficult problem, but in the set Ω matrix $A(\varphi_t(\varphi))$, $\varphi \in \Omega$ may be simpler and easier to investigate.

Generalizing corollaries 1 and 3, it is easy to formulate sufficient conditions for the existence of an asymptotically stable invariant torus of linear extension of dynamical system that has a simple structure of limit sets and recurrent trajectories.

Corollary 4. *Let non-wandering set* Ω *of dynamical system* $\frac{d\varphi}{dt} = a(\varphi)$, $\varphi \in \mathcal{T}_m$ *consist only from the finite number of stationery points* $\{\overline{\varphi}_1, \dots, \overline{\varphi}_k\} = \Phi$ *and finite number of cycles* $\{\mathbb{C}_1, \dots, \mathbb{C}_l\}$ *and the real parts of all eigenvalues of matrices* $A(\varphi)$, $\varphi \in \Phi$ *be negative and all the multipliers of linear periodic systems* $\frac{dx}{dt} = A(\varphi_t(\varphi))x$, $\varphi \in \mathbb{C}_i$, $i = 1, \dots, l$ *lie inside the unit circle. Then system* (7) *has an asymptotically stable invariant toroidal manifold for an arbitrary function* $f(\varphi) \in C(\mathcal{T}_m)$.

The proof of theorem 1 is sufficiently flexible. Utilizing the inequalities of Gronwall–Bellman type one can get similar results for equations of different types, for instance for impulsive differential equations [7, 5]. In papers [3, 2] the analog of corollary 1 is proved for a linear extension of dynamical system on torus with impulsive perturbations at non-fixed moments.

Theorem and corollaries stated here allow to investigate the behavior of sufficiently wide class of multi-frequency systems effectively. Linear extensions of dynamical systems on torus that have a simple structure of non-wandering set are suitable for qualitative analysis without finding fundamental matrices.

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