A Priori Estimates of the Kneser Solutions of Singular in Time and Phase Variables Second Order Differential Inequalities

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In a positive semi-axis $]0,+\infty[$, we consider the second order differential inequalities

$$u''(t) \geq \frac{p(t)}{q(u(t))} \tag{1}$$

and

$$\frac{p(t)}{q(u(t))} \leq u''(t) \leq \frac{p_0(t)}{q_0(u(t))}\,, \tag{2}$$

where $p,\ p_0\ :]0,+\infty[\to [0,+\infty[$ are measurable and $q,\ q_0\ :\ [0,+\infty[\to [0,+\infty[$ are continuous nondecreasing functions such that

$$0 < \int\limits_t^{+\infty} (s-t)p(s)\,ds \leq \int\limits_t^{+\infty} (s-t)p_0(s)\,ds < +\infty \quad \text{for } \ t \geq 0,$$

$$q_0(x) \geq q(x) > 0 \quad \text{for } \ x > 0.$$

A nonincreasing function $u : [0,+\infty[\to]0,+\infty[$ is said to be the Kneser solution of the differential inequality (1) (of the differential inequality (2)) if it is absolutely continuous together with u' on every finite interval contained in $[0,+\infty[$, and satisfies this differential inequality almost everywhere on $]0,+\infty[$.

The Kneser solution u of the differential inequality (1) or (2) is said to be vanishing at infinity if

$$\lim_{t\to+\infty} u(t) = 0.$$

Let

$$Q(x) = \int\limits_0^x q(y)\,dy \quad \text{for } \ x > 0,$$

and let Q^{-1} be the inverse function to Q. Suppose

$$r(t) = Q^{-1}\bigg(\int\limits_t^{+\infty} (s-t)q(s)\,ds\bigg) \quad \text{for } \ t \geq 0.$$

The following theorems are proved.

Theorem 1. *Every Kneser solution of the differential inequality* (1) *admits the estimate*

$$u(t) \geq r(t) \quad \textit{for } \ t \geq 0.$$

Theorem 2. *If*

$$r_0(t) = \int\limits_t^{+\infty} (s-t)\, \frac{p_0(s)}{q_0(r(s))}\, ds < +\infty \quad \textit{for } t \geq 0,$$

then every vanishing at infinity Kneser solution of the differential inequality (2) *admits the estimates*

$$r(t) \leq u(t) \leq r_0(t) \quad \textit{for } t \geq 0.$$

Note that the above formulated theorems cover the case, where $q(0) = q_0(0) = 0$ and

$$\int\limits_0^t p(s)\, ds = +\infty, \quad \int\limits_0^t p_0(s)\, ds = +\infty \quad \text{for } t > 0,$$

i.e. the case, where the differential inequalities (1) and (2) have singularities both in time and phase variables.

Acknowledgements

Supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).