On Kneser Solutions of Second Order Nonlinear Singular Differential Equations

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We consider the problem on the existence of a solution of the differential equation

$$u'' = f(t, u, u'), \tag{1}$$

satisfying the conditions

$$u(0) = c, \quad u(t) > 0, \quad u'(t) < 0 \ \text{ for } \ t > 0. \tag{2}$$

Here $f : D \to R_+$ is a continuous function,

$$D = \big\{(t, x, y) : \ t > 0, \ x > 0, \ y < 0\big\}, \quad R_+ = [0, +\infty[\,,$$

and c is a positive constant.

This problem is often called the Kneser problem since it was first studied by Kneser in the case where $f(t, x, y) \equiv f_0(t, x)$ and $f_0 : R_+ \times R_+ \to R_+$ is a continuous function such that $f_0(t, 0) \equiv 0$.

We are interested in the case where the inequality

$$g_0(t) \le x^\lambda |y|^\mu f(t, x, y) \le g_1(t) \tag{3}$$

is satisfied in the domain D, where λ and μ are positive constants, and $g_i : (0, +\infty) \to (0, +\infty)$ $(i = 0, 1)$ are continuous functions. In this case

$$\lim_{x\to 0} f(t, x, y) = +\infty, \quad \lim_{y\to 0} f(t, x, y) = +\infty,$$

i.e. equation (1) has singularities in phase variables.

The Kneser problem for differential equations with a singularity in one of the phase variables first was investigated by I. Kiguradze [1]. However, in [1] the Kneser problem is considered not for general but for the Emden–Fowler type higher order differential equation

$$u^{(n)} = p(t)u^{-\lambda}.$$

A continuous function $u : [0, +\infty) \to (0, +\infty)$ is said to be the Kneser solution of equation (1) if it is twice continuously differentiable on the open interval $(0, +\infty)$ and satisfies the inequality

$$u'(t) < 0$$

and equation (1) on that interval.

The Kneser solution is called vanishing at infinity if

$$\lim_{t\to+\infty} u(t) = 0,$$

and it is called remote from zero if

$$\lim_{t\to+\infty} u(t) > 0.$$

The following theorems are valid.

Theorem 1. *Let inequality* (3) *be fulfilled. Then for the existence of at least one Kneser solution of equation* (1) *it is necessary the conditions*

$$\int_t^{+\infty} g_0(s)\,ds < +\infty \text{ for } t > 0, \quad \int_0^{+\infty} \bigg(\int_t^{+\infty} g_0(s)\,ds \bigg)^{\frac{1}{\mu+1}} dt < +\infty \tag{4}$$

to be fulfilled. In addition, if conditions (4) *hold, then for any Kneser solution of equation* (1) *the estimate*

$$u(t) > v_0(t;\delta) \ \text{ for } \ t \geq 0$$

is valid, where

$$v_0(t;\delta) = \bigg[\delta^\nu + (1+\mu)^{\frac{1}{1+\mu}} \nu \int_t^{+\infty} \bigg(\int_s^{+\infty} g_0(x)\,dx \bigg)^{\frac{1}{1+\mu}} ds \bigg]^{\frac{1}{\nu}},$$

$$\delta = u(+\infty), \quad \nu = \frac{1+\nu+\mu}{1+\mu}\,.$$

Theorem 2. *If*

$$\int_t^{+\infty} g_1(s)\,ds < +\infty \text{ for } t > 0, \quad \int_0^{+\infty} \bigg(\int_t^{+\infty} g_1(s)\,ds \bigg)^{\frac{1}{\mu+1}} dt < +\infty, \tag{5}$$

then for each positive number δ equation (1) *has at least one Kneser solution satisfying the equality*

$$\lim_{t \to +\infty} u(t) = \delta. \tag{6}$$

Theorem 3. *If along with* (3) *and* (5) *the conditions*

$$\int_t^{+\infty} \frac{g_1(s)}{v_0^\lambda(s,0)}\,ds < +\infty \ \text{ for } \ t > 0, \quad \int_0^{+\infty} \bigg(\int_t^{+\infty} \frac{g_1(s)}{v_0^\lambda(s,0)}\,ds \bigg)^{\frac{1}{\mu+1}} dt < +\infty$$

are satisfied, then equation (1) *has at least one vanishing at infinity Kneser solution.*

Suppose conditions (5) hold. We introduce the function

$$v_1(t;\delta) = \delta + \bigg[(1+\mu) \int_t^{+\infty} \bigg(\int_s^{+\infty} \frac{g_1(x)}{v_0^\lambda(x;\delta)}\,dx \bigg)^{\frac{1}{\mu+1}} ds \bigg], \ \ t \geq 0, \ \ \delta > 0,$$

and the constant

$$c_0 = \inf \big\{ v_1(0;\delta) : \ \delta > 0 \big\}.$$

Theorem 4. *Let conditions* (3) *and* (5) *be fulfilled. If, moreover,*

$$c > c_0,$$

then problem (1), (2) *has at least one solution, and if*

$$c \leq v_0(0;0),$$

then problem (1), (2) *has no solution.*

Remark. In the case where conditions (5) hold and

$$c \in (v_0(0;0), c_0],$$

then the question on the solvability of problem (1), (2) remains open.

Consider now the case, where $g_1(t) \equiv \ell g_0(t)$, $\ell = const \geq 1$, i.e. the case where inequality (3) has the form

$$g_0(t) \leq x^\lambda |y|^\mu f(t,x,y) \leq \ell g_0(t), \tag{7}$$

where, as above, $g_0 : (0,+\infty) \to (0,+\infty)$ is a continuous function and λ, μ are positive constants.

From Theorems 1, 2, and 4 it follows

Corollary 1. *Let inequality* (7) *be fulfilled. Then the following assertions are equivalent:*

(A) *Conditions* (4) *are satisfied;*

(B) *Equation* (1) *has at least one remote from zero Kneser solution;*

(C) *Problem* (1), (6) *is solvable for each positive number* δ*;*

(D) *Problem* (1), (2) *is solvable for large* c*.*

Corollary 2. *Let inequality* (7) *be fulfilled, where*

$$g_0(t) = \begin{cases} \gamma\, t^{-\alpha} & \textit{for } 0 < t \leq 1, \\ \gamma\, t^{-\beta} & \textit{for } t > 1, \end{cases}$$

$\gamma > 0$, $\alpha = const$, $\beta = const$. *Then the following assertions are equivalent:*

(A) $\alpha < 2 + \mu$, $\beta > 2 + \mu$;

(B) *Equation* (1) *has at least one remote from zero Kneser solution;*

(C) *Equation* (1) *has at least one vanishing at infinity Kneser solution;*

(D) *Problem* (1), (6) *is solvable for each positive number* δ*;*

(E) *Problem* (1), (2) *is solvable for large* c*.*

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References

[1] I. Kiguradze, On Kneser solutions to the Emden-Fowler differential equation with a negative exponent. *Tr. Inst. Mat., Minsk* **4** (2000), 69–77.