Myshkis Type Oscillation Criteria for Second-Order Linear Delay Differential Equations

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On the half-line $\mathbb{R}_+ = [0, +\infty[$ we consider the second-order differential equation

$$u''(t) + p(t)u(\tau(t)) = 0, \tag{1}$$

where $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lebesgue integrable function and $\tau\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that

$$\tau(t) \le t \quad \text{for } \ t \ge 0, \qquad \lim_{t\to+\infty} \tau(t) = +\infty.$$

Solutions to equation (1) can be defined in various ways. Since we are interested in properties of solutions in a neighbourhood of $+\infty$, we introduce the following commonly used definition.

Definition 1. Let $t_0 \in \mathbb{R}_+$ and $a_0 = \min\{\tau(t) : \ t \ge t_0\}$. A continuous function $u\colon [a_0, +\infty[\to \mathbb{R}$ is said to be a *solution to equation* (1) *on the interval* $[t_0, +\infty[$ if it is absolutely continuous together with its first derivative on every compact interval contained in $[t_0, +\infty[$ and satisfies equality (1) almost everywhere in $[t_0, +\infty[$. A solution u to equation (1) on the interval $[t_0, +\infty[$ is called *proper* if the inequality $\sup\big\{|u(s)| : \ s \ge t\big\} > 0$ holds for $t \ge t_0$.

Definition 2. A proper solution to equation (1) is said to be *oscillatory* if it has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

We have proved in [3, Proposition 2.1] that if $\int\limits_0^{+\infty} sp(s)\,\mathrm{d}s < +\infty$, then (1) has a proper non-oscillatory solution. Therefore, we assume in what follows that $\int\limits_0^{+\infty} sp(s)\,\mathrm{d}s = +\infty$. Moreover, we consider the case where

$$\int\limits_0^{+\infty} \frac{\tau(s)}{s}\, p(s)\,\mathrm{d}s < +\infty \tag{2}$$

and

$$\limsup_{t\to+\infty} \frac{1}{t} \int\limits_0^t s\tau(s)p(s)\,\mathrm{d}s \le 1, \quad \limsup_{t\to+\infty} t \int\limits_t^{+\infty} \frac{\tau(s)}{s}\, p(s)\,\mathrm{d}s \le 1, \tag{3}$$

because otherwise every proper solution to equation (1) is oscillatory (see, e. g., [2, Corollaries 3.4 and 3.5]).

In the paper [1], R. Koplatadze proved, among other things, the following oscillation criteria.

Criterion 1 ([1, Theorem 1])**.** *Let there exist a continuous non-decreasing function* $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that the inequalities*

$$\tau(t) \le \sigma(t) \le t \quad \textit{for } t \ge 0 \tag{4}$$

and

$$\limsup_{t\to+\infty} \int_{\sigma(t)}^{t} \tau(s)p(s)\,\mathrm{d}s > 1 \tag{5}$$

are fulfilled. Then every proper solution to equation (1) *is oscillatory.*

Criterion 2 ([1, Theorem 2])**.** *Let the inequality*

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} \tau(s)p(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{6}$$

hold. Then every proper solution to equation (1) *is oscillatory.*

Remark 1. In Criterion 2, the constant $\frac{1}{\mathrm{e}}$ is optimal and can not be in general improved. A counterexample is constructed in [1] for equation (1) with a proportional delay.

Below we present new Myshkis type oscillation criteria for equation (1), which generalise known results of R. Koplatadze. In particular, under some natural additional assumptions, we improve constants on the right-hand side of inequalities (5) and (6).

Let

$$G_* := \liminf_{t\to+\infty} \frac{1}{t} \int_0^t s\tau(s)p(s)\,\mathrm{d}s, \quad F_* := \liminf_{t\to+\infty} t \int_t^{+\infty} \frac{\tau(s)}{s}\, p(s)\,\mathrm{d}s.$$

In view of assumption (2), the number F_* is well defined and, moreover, assumptions (3) yield that $G_* \le 1$ and $F_* \le 1$. We shall also assume in what follows that

$$\int_0^{+\infty} s^{\lambda}\Big(\frac{\tau(s)}{s}\Big)^{1-\varepsilon G_*} p(s)\,\mathrm{d}s < +\infty \quad \text{for all } \lambda < 1,\ \ \varepsilon \in [0,1[\,, \tag{7}$$

because otherwise every proper solution to equation (1) is oscillatory without any additional condition (see [3, Theorem 2.4]). It allows one to define, for any $\lambda < 1$ and $\varepsilon \in [0,1[$, the function

$$Q(t;\lambda,\varepsilon) := t^{1-\lambda} \int_t^{+\infty} s^{\lambda}\Big(\frac{\tau(s)}{s}\Big)^{1-\varepsilon G_*} p(s)\,\mathrm{d}s \quad \text{for } t > 0.$$

Moreover, for any $\mu > 1$ and $\varepsilon \in [0,1[$, we put

$$H(t;\mu,\varepsilon) := \frac{1}{t^{\mu-1}} \int_0^t s^{\mu}\Big(\frac{\tau(s)}{s}\Big)^{1-\varepsilon G_*} p(s)\,\mathrm{d}s \quad \text{for } t > 0.$$

Corollaries 2.11 and 2.12 stated in [3] claim that every proper solution to equation (1) is oscillatory provided that for some $\lambda < 1$, $\mu > 1$, and $\varepsilon \in [0,1[$,

$$\text{either } Q_*(\lambda,\varepsilon) > \frac{1}{4(1-\lambda)} \text{ or } H_*(\mu,\varepsilon) > \frac{1}{4(\mu-1)}\,,$$

where

$$Q_*(\lambda,\varepsilon) := \liminf_{t\to+\infty} Q(t;\lambda,\varepsilon), \quad H_*(\mu,\varepsilon) := \liminf_{t\to+\infty} H(t;\mu,\varepsilon).$$

Therefore, it is natural to restrict ourself to the case where

$$Q_*(\lambda,\varepsilon) \le \frac{1}{4(1-\lambda)}, \quad H_*(\mu,\varepsilon) \le \frac{1}{4(\mu-1)} \quad \text{for all } \lambda<1,\ \mu<1,\ \varepsilon\in[0,1[\,.$$

Under these assumptions, we can improve Criteria 1 and 2, for example, as follows.

Theorem 1. *Let there exist numbers $\lambda<1$, $\mu>1$, $\varepsilon\in[0,1[$ and a non-decreasing function $\sigma\colon \mathbb{R}_+\to\mathbb{R}_+$ such that condition* (4) *is fulfilled,*

$$\frac{\lambda(2-\lambda)}{4(1-\lambda)} \le Q_*(\lambda,\varepsilon) \le \frac{1}{4(1-\lambda)}\,, \quad \frac{\mu(2-\mu)}{4(\mu-1)} \le H_*(\mu,\varepsilon) \le \frac{1}{4(\mu-1)}\,, \tag{8}$$

and

$$\limsup_{t\to+\infty} \int\limits_{\sigma(t)}^{t} \tau(s)p(s)\Big(\frac{\sigma(s)}{\tau(s)}\Big)^{\varepsilon G_*}\,ds > R_0-\alpha_* r_0,$$

where

$$r_0 := \frac{1}{2}\Big(1-\sqrt{1-4(1-\lambda)Q_*(\lambda,\varepsilon)}\,\Big), \quad R_0 := \frac{1}{2}\Big(1+\sqrt{1-4(\mu-1)H_*(\mu,\varepsilon)}\,\Big), \tag{9}$$

and $\alpha_* := \liminf\limits_{t\to+\infty} (\frac{\sigma(t)}{t})^{1-\varepsilon F_*}$. Then every proper solution to equation (1) *is oscillatory.*

Remark 2. Observe that $0\le\alpha_*\le 1$ and $\max\{\frac{\lambda}{2},0\}\le r_0\le\frac{1}{2}\le R_0\le\min\{\frac{\mu}{2},1\}$. Therefore, we have $R_0-\beta_* r_0\le 1$ and thus Theorem 1 improves (under additional assumptions (8)) Criterion 1.

Theorem 2. *Let there exist numbers $\lambda<1$, $\mu>1$, and $\varepsilon\in[0,1[$ such that inequalities* (8) *are satisfied and*

$$\liminf_{t\to+\infty} \frac{t}{\tau(t)} < +\infty, \quad \liminf_{t\to+\infty} \tau^{\varepsilon G_*}(t) \int\limits_{\tau(t)}^{t} \tau^{1-\varepsilon G_*}(s)p(s)\,ds > R_0-\beta_* r_0,$$

where the numbers r_0 and R_0 are given by relations (9) *and $\beta_* := \liminf\limits_{t\to+\infty} (\frac{\tau(t)}{t})^{\varepsilon G_*}$. Then every proper solution to equation* (1) *is oscillatory.*

Remark 3. Observe that $0\le\beta_*\le 1$, the numbers r_0 and R_0 given by relations (9) satisfy

$$R_0-r_0 = \frac{1}{2}\Big(\sqrt{1-4(1-\lambda)Q_*(\lambda,\varepsilon)}+\sqrt{1-4(\mu-1)H_*(\mu,\varepsilon)}\,\Big),$$

and thus the difference R_0-r_0 converges to zero if $Q_*(\lambda,\varepsilon)\to\frac{1}{4(1-\lambda)}$ and $H_*(\mu,\varepsilon)\to\frac{1}{4(\mu-1)}$. Consequently, it may happen that $R_0-\beta_* r_0<\frac{1}{\mathrm{e}}$ in which case Theorem 2 improves Criterion 2.

References

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