Nonlocal Boundary Value Problem for Strongly Singular Higher-Order Linear Functional-Differential Equations

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Consider the differential equations with deviating arguments

$$u^{(2m+1)}(t)=\sum_{j=0}^{m}p_j(t)u^{(j)}(\tau_j(t))+q(t)\quad\text{for }\ a<t<b, \tag{1}$$

with the boundary conditions

$$\begin{gathered}\int\limits_a^b u(s)\,d\varphi(s)=0\quad\text{where }\ \varphi(b)-\varphi(a)\neq 0,\\ u^{(i)}(a)=0,\quad u^{(i)}(b)=0\ \ (i=1,\dots,m).\end{gathered} \tag{2}$$

Here $m\in N$, $-\infty<a<b<+\infty$, $p_j,q\in L_{loc}(]a,b[)$ $(j=0,\dots,m)$, $\varphi:[a,b]\to R$ is a function of bounded variation, and $\tau_j:]a,b[\,\to\,]a,b[$ are measurable functions. By $u^{(i)}(a)$ (resp., $u^{(i)}(b)$), we denote the right (resp., left) limit of the function $u^{(i)}$ at the point a (resp., b).

The Agarwal–Kiguradze type theorems [1] are obtained by us, which contains unimprovable in a certain sense conditions guaranteeing the unique solvability of problem (1), (2).

We use the following notations.

$R^+=[0,+\infty[\,$;

$[x]_+$ is the positive part of a number x, that is $[x]_+=\frac{x+|x|}{2}$;

$L_{loc}(]a,b[)$ is the space of functions $y:]a,b[\,\to R$, which are integrable on $[a+\varepsilon,b-\varepsilon]$ for arbitrary small $\varepsilon>0$;

$L_{\alpha,\beta}(]a,b[)$ ($L^2_{\alpha,\beta}(]a,b[)$) is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y:]a,b[\,\to R$, with the norm

$$\|y\|_{L_{\alpha,\beta}}=\int\limits_a^b(s-a)^\alpha(b-s)^\beta|y(s)|\,ds\quad\bigg(|y\|_{L^2_{\alpha,\beta}}=\bigg(\int\limits_a^b(s-a)^\alpha(b-s)^\beta y^2(s)\,ds\bigg)^{1/2}\bigg);$$

$L([a,b])=L_{0,0}(]a,b[)$, $L^2([a,b])=L^2_{0,0}(]a,b[)$;

$M(]a,b[)$ is the set of measurable functions $\tau:]a,b[\,\to\,]a,b[\,$;

$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ is the Banach space of functions $y\in L_{loc}(]a,b[)$ such that

$$\|y\|_{\widetilde{L}^2_{\alpha,\beta}}:=\max\bigg\{\bigg[\int\limits_a^t(s-a)^\alpha\bigg(\int\limits_s^t y(\xi)\,d\xi\bigg)^2ds\bigg]^{1/2}:\ a\le t\le\frac{a+b}{2}\bigg\}+\\ +\max\bigg\{\bigg[\int\limits_t^b(b-s)^\beta\bigg(\int\limits_t^s y(\xi)\,d\xi\bigg)^2ds\bigg]^{1/2}:\ \frac{a+b}{2}\le t\le b\bigg\}<+\infty.$$

$\widetilde{C}^n_{loc}(]a,b[)$ is the space of functions $y :]a,b[\to R$ which are absolutely continuous together with $y', y'', \dots, y^{(n)}$ on $[a+\varepsilon, b-\varepsilon]$ for an arbitrarily small $\varepsilon > 0$.

$\widetilde{C}^{\,n,m}(]a,b[)$ $(m \le n)$ is the space of functions $y \in \widetilde{C}^n_{loc}(]a,b[)$, satisfying

$$\int\limits_a^b |y^{(m)}(s)|^2\, ds < +\infty. \tag{3}$$

When problem (1), (2) is discussed, we assume that the conditions

$$p_j \in L_{loc}(]a,b[) \ \ (j = 0,\dots,m) \tag{4}$$

are fulfilled.

A solution of problem (1), (2) is sought for in the space $\widetilde{C}^{\,2m,m+1}(]a,b[)$.

By $h_j :]a,b[\times]a,b[\to R_+$ and $f_j : R \times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ $(j = 1,\dots,m)$ we denote the functions and, respectively, the operators defined by the equalities

$$\begin{aligned} h_1(t,s) &= \bigg|\int\limits_s^t \big[(-1)^m p_1(\xi)\big]_+\, d\xi\bigg|,\\ h_j(t,s) &= \bigg|\int\limits_s^t p_j(\xi)\, d\xi\bigg| \ \ (j = 2,\dots,m), \end{aligned} \tag{5}$$

and,

$$f_j(c,\tau_j)(t,s) = \bigg|\int\limits_s^t |p_j(\xi)|\,\bigg|\int\limits_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)}\, d\xi_1\bigg|^{1/2} d\xi\bigg| \ \ (j = 1,\dots,m), \tag{6}$$

and also we put that

$$f_0(t,s) = \bigg|\int\limits_s^t |p_0(\xi)|\, d\xi\bigg|.$$

Let $m = 2k+1$, then

$$m!! = \begin{cases} 1 & \text{for } \ m \le 0 \\ 1\cdot 3\cdot 5\cdots m & \text{for } \ m \ge 1 \end{cases}.$$

Along with (1), we consider the homogeneous equation

$$v^{(2m+1)}(t) = \sum_{j=0}^m p_j(t) v^{(j)}(\tau_j(t)) \quad \text{for} \quad a < t < b. \tag{1_0}$$

Theorem 1. *Let there exist $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, numbers $l_{kj} > 0$, $\gamma_{k0} > 0$, $\gamma_{kj} > 0$ $(k = 0,1;$ $j = 1,\dots,m)$ such that*

$$\begin{gathered} (t-a)^{2m-j} h_j(t,s) \le l_{0j} \quad (j = 1,\dots,m) \ \ \textit{for} \ \ a < t \le s \le a_0,\\ \limsup_{t\to a}\, (t-a)^{m-\frac12-\gamma_{00}} f_0(t,s) < +\infty,\\ \limsup_{t\to a}\, (t-a)^{m-\frac12-\gamma_{0j}} f_j(a,\tau_j)(t,s) < +\infty \ \ (j = 1,\dots,m), \end{gathered} \tag{7}$$

$$\begin{gathered} (b-t)^{2m-j} h_j(t,s) \le l_{1j} \quad (j = 1,\dots,m) \ \ \textit{for} \ \ b_0 \le s \le t < b,\\ \limsup_{t\to b}\, (b-t)^{m-\frac12-\gamma_{10}} f_0(t,s) < +\infty,\\ \limsup_{t\to b}\, (b-t)^{m-\frac12-\gamma_{1j}} f_j(b,\tau_j)(t,s) < +\infty \ \ (j = 1,\dots,m), \end{gathered} \tag{8}$$

and

$$\sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\, l_{kj} < 1 \ \ (k=0,1). \tag{9}$$

Let, moreover, the homogeneous problem $(1_0),(2)$ *have only the trivial solution in the space* $\widetilde{C}^{\,2m,m+1}(]a,b[)$. *Then problem* $(1),(2)$ *has a unique solution* u *for an arbitrary* $q\in\widetilde{L}^2_{2m-2,2m-2}(]a,b[)$, *and there exists a constant* r, *independent of* q, *such that*

$$\|u^{(m+1)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2m-2,2m-2}}. \tag{10}$$

Theorem 2. *Let there exist numbers* $t^*\in\,]a,b[\,$, $l_{k0}>0$, $l_{kj}>0$, $\bar{l}_{kj}\ge 0$, *and* $\gamma_{k0}>0$, $\gamma_{kj}>0$ $(k=0,1;\ j=1,\dots,m)$ *such that along with*

$$B_0 \equiv \bar{l}_{00}\Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}}\,\frac{(t^*-a)^{\gamma_{00}}}{\sqrt{2\gamma_{00}}}\int_a^b \frac{|\varphi(\xi)-\varphi(a)|+|\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|}\,d\xi + $$
$$+\sum_{j=1}^{m}\Big(\frac{(2m-j)2^{2m-j+1}l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\,\bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}}\Big) < \frac{1}{2}\,, \tag{11}$$

$$B_1 \equiv \bar{l}_{10}\Big(\frac{2^{m-1}}{(2m-3)!!}\Big)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}}\,\frac{(b-t^*)^{\gamma_{10}}}{\sqrt{2\gamma_{10}}}\int_a^b \frac{|\varphi(\xi)-\varphi(a)|+|\varphi(\xi)-\varphi(b)|}{|\varphi(b)-\varphi(a)|}\,d\xi + $$
$$+\sum_{j=1}^{m}\Big(\frac{(2m-j)2^{2m-j+1}l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\,\bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}}\Big) < \frac{1}{2}\,, \tag{12}$$

the conditions

$$\begin{gathered}(t-a)^{m-\gamma_{00}-1/2} f_0(t,s) \le \bar{l}_{00},\\ (t-a)^{2m-j} h_j(t,s) \le l_{0j}, \quad (t-a)^{m-\gamma_{0j}-1/2} f_j(a,\tau_j)(t,s) \le \bar{l}_{0j}\end{gathered} \tag{13}$$

for $a<t\le s\le t^*$ *and*

$$\begin{gathered}(b-t)^{m-\gamma_{10}-1/2} f_0(t,s) \le \bar{l}_{10},\\ (b-t)^{2m-j} h_j(t,s) \le l_{1j}, \quad (b-t)^{m-\gamma_{1j}-1/2} f_j(b,\tau_j)(t,s) \le \bar{l}_{1j}\end{gathered} \tag{14}$$

for $t^*\le s\le t<b$ *hold with any* $j=1,\dots,m$. *Then problem* $(1),(2)$ *is uniquely solvable in the space* $\widetilde{C}^{\,2m,m+1}(]a,b[)$ *for every* $q\in\widetilde{L}^2_{2m-2,2m-2}(]a,b[)$.

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References

[1] R. P. Agarwal and I. Kiguradze, Two-point boundary value problems for higher-order linear differential equations with strong singularities. *Bound. Value Probl.* **2006**, Art. ID 83910, 32 pp.