Generalized Linear Differential Equations in a Banach Space (Extension of the Opial Continuous Dependence Result)

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Our aim is to present new conditions ensuring the continuous dependence on a parameter k of solutions to linear integral equations of the form

$$x(t)=\widetilde{x}_k+\int_a^t \mathrm{d}[A_k]\,x+f_k(t)-f_k(a),\ \ t\in[a,b], k\in\mathbb{N}\,, \qquad (eq_k)$$

where $-\infty<a<b<\infty$, X is a Banach space, $L(X)$ is the Banach space of linear bounded operators on X, $\widetilde{x}_k\in X$, $A_k:[a,b]\to L(X)$ have bounded variations on $[a,b]$, $f_k:[a,b]\to X$ are regulated on $[a,b]$. The integrals are understood as the abstract Kurzweil–Stieltjes integrals and the studied equations are usually called generalized linear differential equations (in the sense of J. Kurzweil, cf. [3] or [4]). Basic results on the theory of Kurzweil–Stieltjes integral in abstract spaces can be found, for example, in [5] and [10].

Continuing in our research from [6], here we focus our attention on the case when the variations $\operatorname{var}_a^b A_k$ need not be uniformly bounded. More precisely, here we extend Theorem 4.2 from [6], which is an analogy of the Opial's result [7] for ODEs. The new result reads as follows:

Main Theorem. *Assume:* $A_k\in\mathrm{BV}([a,b],L(X))$, $f_k\in G([a,b],X)$, $\widetilde{x}_k\in X$ *for* $k\in\mathbb{N}$,

- $A\in\mathrm{BV}([a,b],L(X))$, $f\in\mathrm{BV}([a,b],X)$, $\widetilde{x}\in X$,
- $\big[I-\Delta^-A(t)\big]^{-1}\in L(X)$ *for* $t\in(a,b]$,
- $\widetilde{x}_k\to\widetilde{x}$,
- $\lim\limits_{k\to\infty}\big(1+\operatorname{var}_a^b A_k\big)\|A_k-A\|_\infty=0$,
- $\lim\limits_{k\to\infty}\big(1+\operatorname{var}_a^b A_k\big)\|f_k-f\|_\infty=0$.

Then the equation

$$x(t)=\widetilde{x}+\int_a^t \mathrm{d}[A]x+f(t)-f(a),\ \ t\in[a,b], \qquad (\mathrm{eq})$$

has a unique solution $x\in\mathrm{BV}([a,b],X)$, *equation* (eq_k) *has a unique solution and* $x_k\in G([a,b],X)$ *for* $k\in\mathbb{N}$ *sufficiently large and* $x_k\rightrightarrows x$ *on* $[a,b]$.

The proof relies on the following lemma which is an analogue of the assertion formulated for ODEs by Kiguradze in [2, Lemma 2.5]. Its variant was used also for FDEs by Hakl, Lomtatidze and Stavrolaukis in [1, Lemma 3.5].

Lemma. *Assume:* $A_k\in\mathrm{BV}([a,b],L(X))$ *for* $k\in\mathbb{N}$,

- $A\in\mathrm{BV}([a,b],L(X))$ *with* $\big[I-\Delta^-A(t)\big]^{-1}\in L(X)$ *for* $t\in(a,b]$,
- $\lim\limits_{k\to\infty}\big(1+\operatorname{var}_a^b A_k\big)\|A_k-A\|_\infty=0$.

Then there exist $r^*>0$ *and* $k_0 \in \mathbb{N}$ *such that*

$$\|y\|_\infty \le r^*\bigg(\|y(a)\|_X + \big(1+\operatorname{var}_a^b A_k\big) \sup_{t\in[a,b]} \Big\| y(t)-y(a)-\int_a^t \mathrm{d}[A_k]y\Big\|_X\bigg)$$

for all $y \in G([a,b],X)$ *and* $k \ge k_0$.

In the sequel, we present the example that shows that the condition $f \in \mathrm{BV}([a,b],X)$ in Main Theorem can not be avoided.

Example. Let $[a,b]=[0,1]$. For $k \in \mathbb{N}$ put

$$n_k = [k^{3/2}]+1, \quad \tau_{m,k} = \frac{1}{2^{n_k-m}} \quad \text{for } m \in \{0,1,\dots,n_k\}$$

(where $[\lambda]$ stands, as usual, for the integer part of the nonnegative real number λ),

$$\begin{aligned} a_{0,k} &= (-1)^{n_k}\,\frac{2^{n_k}}{k}\,, \quad b_{0,k} = (-1)^{n_k-1}\,\frac{1}{k}\,,\\ a_{m,k} &= (-1)^{n_k-m}\,\frac{2^{n_k-m+1}}{k}\,, \quad b_{m,k} = (-1)^{n_k-m+1}\,\frac{3}{k} \quad \text{for } m \in \{1,2,\dots,n_k-1\}, \end{aligned}$$

and define

$$A_k(t) = \begin{cases} 0 & \text{for } t \in [0,\tau_0^k],\\ a_{m,k}t+b_{m,k} & \text{for } t \in [\tau_{m,k},\tau_{m+1,k}] \text{ and } m \in \{0,1,\dots,n_k-1\}, \end{cases}$$
$$A(t) = 0 \quad \text{for } t \in [0,1].$$

We can verify that $\operatorname{var}_0^1 A_k<\infty$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \big(1+\operatorname{var}_a^b A_k\big)\|A_k-A\|_\infty \le \lim_{k\to\infty}\Big(\frac{1}{k}+\frac{2}{\sqrt{k}}\Big) = 0.$$

Consider the function

$$f(t) = \begin{cases} \dfrac{(-1)^k}{\sqrt[4]{n}} & \text{if } t \in (2^{-n},2^{-(n-1)}] \text{ for some } n \in \mathbb{N},\\ 0 & \text{if } t=0, \end{cases}$$

and define $f_k(t)=f(t)$ for $t \in [0,1]$ and $k \in \mathbb{N}$. Note that, the conditions of the Main Theorem are satisfied, except for the fact that $\operatorname{var}_0^1 f=\infty$. However, it is possible to prove that the sequence of solutions x_k of (eq_k) does not converge to the solution of (eq). Roughly speaking, we can observe that, for each $k \in \mathbb{N}$, $x_k(1)$ involves partial sums of divergent series $\sum_{m=2}^{\infty} \frac{1}{\sqrt[4]{m}}$. Therefore, $x_k(1)$ cannot have a finite limit for $k\to\infty$.

Applications to linear dynamic equations on time scales are then enabled by their relationship with generalized differential equations as disclosed by A. Slavík in [8].

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