On the Solvability of Linear Overdetermined Boundary Value Problems for a Class of Functional Differential Equations

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1 Introduction

We consider here a system of functional differential equations (FDE, FDS) that, formally speaking, is a concrete realization of the so-called abstract functional differential equation (AFDE). Theory of AFDE is thoroughly treated in [1, 4, 3]. On the other hand, the system under consideration covers many kinds of dynamic models with aftereffect (integro-differential, delayed differential, differential difference, difference) [6, 7, 8].

First we descript in detail a class of functional differential equations with linear Volterra operators and appropriate spaces where those are considered. We concerned with the representation of general solution and basic relationships for the Cauchy operator. Next the setting of the general linear boundary value problem (BVP) is given and discussed, and conditions for the solvability of BVP are obtained in the case that it is not everywhere solvable. As for linear overdetermined BVP's for AFDE in general, the principal results by L. F. Rakhmatullina are given in detail in [1, 4, 3]. Here we propose a somewhat different approach without recourse to the adjoint BVP and an extension of the original BVP. Our approach is based, in essence, on the assumption that the space to which the derivative of the solution does belong is a Hilbert space.

Let us introduce the Banach spaces where operators and equations are considered.

Fix a segment $[0,T] \subset R$. By $L_2^n = L_2^n[0,T]$ we denote the Hilbert space of square summable functions $v : [0,T] \to R^n$ with the inner product $(u,v) = \int\limits_0^T u'(t)v(t)\,dt$ ($\cdot'$ is the symbol of transposition). The space $AC_2^n = AC_2^n[0,T]$ is the space of absolutely continuous functions $x : [0,T] \to R^n$ such that $\dot{x} \in L_2^n$ with the norm $\|x\|_{AC_2^n} = |x(0)| + \sqrt{(\dot{x},\dot{x})}$, where $|\cdot|$ stands for the norm of R^n.

In what follows we will use some results from [2, 5, 4] concerning the equation

$$\mathcal{L}x \equiv \dot{x} - \mathcal{K}\dot{x} - A(\cdot)x(0) = f, \tag{1}$$

where the linear bounded operator $\mathcal{K} : L_2^n \to L_2^n$ is defined by $(\mathcal{K}z)(t) = \int\limits_0^t K(t,s)z(s)\,ds$, $t \in [0,T]$, the elements $k_{ij}(t,s)$ of the kernel $K(t,s)$ are measurable on the set $0 \le s \le t \le T$ and such that $|k_{ij}(t,s)| \le u(t)v(s)$, $i,j = 1,\dots,n$, $u,v \in L_2^1[0,T]$, $n \times n$-matrix A has elements square summable on $[0,T]$.

Recall that the homogeneous equation (1) ($f(t) = 0$, $t \in [0,T]$) has the fundamental matrix $X(t)$ of dimension $n \times n$:

$$X(t) = E_n + V(t),$$

where E_n is the identity $n \times n$-matrix, each column $\nu_i(t)$ of the $n \times (n+mn)$-matrix $V(t)$ is a unique solution to the Cauchy problem

$$\dot{\nu}(t) = \int\limits_0^t K(t,s)\dot{\nu}(s)\,ds + a_i(t), \quad \nu(0) = 0, \quad t \in [0,T],$$

$a_i(t)$ is the i-th column of the matrix A.

The solution of (1) with the initial condition $x(0) = 0$ has the representation $x(t) = (Cf)(t) = \int_0^t C(t,s)f(s)\,ds$, where $C(t,s)$ is the Cauchy matrix of the operator $\mathcal{L}$. This matrix can be defined (and constructed) as the solution to

$$\frac{\partial}{\partial t} C(t,s) = \int_s^t K(t,\tau)\,\frac{\partial}{\partial \tau} C(\tau,s)\,d\tau + K(t,s), \;\; 0 \le s \le t \le T,$$

under the condition $C(s,s) = E_n$.

The matrix $C(t,s)$ is expressed in terms of the resolvent kernel $R(t,s)$ of the kernel $K(t,s)$. Namely, $C(t,s) = E_n + \int_s^t R(\tau,s)\,d\tau$. The general solution of (1) has the form $x(t) = X(t)\alpha + \int_0^t C(t,s)f(s)\,ds$, with arbitrary $\alpha \in R^{n+mn}$.

2 General Linear Boundary Value Problem

The general linear BVP is the system (1) supplemented by linear boundary conditions

$$\ell x = \gamma, \;\; \gamma \in R^N, \tag{2}$$

where $\ell : AC^n \to R^N$ is the linear bounded vector functional. Let us recall the representation of ℓ:

$$\ell x = \int_0^T \Phi(s)\dot{x}(s)\,ds + \Psi x(0). \tag{3}$$

Here Ψ is a constant $N \times n$-matrix, Φ is $N \times n$ matrix with square summable on $[0,T]$ elements. We assume that the components $\ell^i : AC_2^n \to R$, $i = 1,\dots,N$ are linearly independent.

BVP (1), (2) is well-defined if $N = n$. In such a situation, BVP (1), (2) is uniquely solvable for any f, γ if and any if the matrix

$$\ell X = (\ell X^1, \dots, \ell X^n),$$

where X^j is the j-th column of X, is nonsingular, i.e. $\det \ell X \neq 0$. It should be noted that this condition cannot be verified immediately because the fundamental matrix X cannot be (as a rule) evaluated explicitly. In addition, even if X were known, then the elements of ℓX, generally speaking, could not be evaluated explicitly. By the theorem about inverse operators, the matrix ℓX is invertible if one can find an invertible matrix Γ such that $\|\ell X - \Gamma\| < 1/\|\Gamma^{-1}\|$. As it has been shown in [9], such a matrix Γ for the invertible matrix ℓX always can be found among the matrices $\Gamma = \overline{\overline{\ell}\,\overline{X}}$, where $\overline{\ell} : AC_2^n \to R^n$ is a vector-functional near to ℓ, and $\overline{X}$ is an approximation of X. That is why the basis of the so-called constructive study of linear BVP's includes a special technique of approximate constructing the solutions to FDE with guaranteed explicit error bounds as well as the reliable computing experiment (RCE), whose theory has been worked out in [7, 9, 8].

We assume in the sequel that $N > n$ and the system $\ell^i : AC_2^n \to R$, $i = 1,\dots,N$ can be splitted into two subsystems $\ell_1 : AC_2^n \to R^n$ and $\ell_2 : AC_2^n \to R^{N-n}$ such that the BVP

$$\mathcal{L}x = f, \;\; \ell_1 x = \gamma_1 \tag{4}$$

is uniquely solvable. Without loss of generality we will consider that ℓ_1 is formed by first n components of ℓ and the elements of γ_1 in (4) are the corresponding components of γ. Thus ℓ_2 will stand for the final $(N-n)$ components of ℓ, and elements of $\gamma_2 \in R^{N-n}$ are defined as the final $(N-n)$ components of γ.

Define the $(N-n)\times n$-matrix $B(s)$ with square summable elements by the representation

$$\ell_2 Cf - (\ell_2 X)(\ell_1 X)^{-1}\ell_1 Cf = \int_0^T B(s)f(s)\,ds \tag{5}$$

for all $f \in L_2^n$. An explicit form of B is simple to derive by elementary transformations taking into account (3) and the properties of the Cauchy matrix $C(t,s)$.

Theorem. *Let the matrix* $W = \int_0^T B(s)B'(s)\,ds$, *where* B *is defined by* (5), *be nonsingular. Then BVP* (1), (2) *is solvable for all* f *of the form*

$$f(t) = f_0(t) + \varphi(t),$$

where

$$f_0(t) = B'(t)\big[W^{-1}\gamma_2 - W^{-1}(\ell_2 X)(\ell_1 X)^{-1}\gamma_1\big],$$

$\varphi(\,\cdot\,) \in L_2^n$ *is arbitrary function orthogonal to each column of* $B'(\,\cdot\,)$: $\int_0^T B(s)\varphi(s)\,ds = 0$.

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