An Axiomatic Definition for Smallness Classes in Lyapunov Exponents Theory

E. K. Makarov

Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: `jcm@im.bas-net.by`

Consider the linear system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with a piecewise continuous and bounded coefficient matrix A. Together with the system (1) consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \quad y \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

with a piecewise continuous and bounded perturbation matrix Q. Denote the Cauchy matrix of (1) by X_A and the higher exponent of (2) – by $\lambda_n(A+Q)$.

One of the basic problem of Lyapunov exponents theory is to evaluate the quantity $\Lambda(\mathfrak{M}) := \sup\{\lambda_n(A+Q) : Q \in \mathfrak{M}\}$ where $\mathfrak{M}$ is a *smallness class* of perturbations (see [1] and the references therein). The notion of smallness class is not precisely defined in general. The following classes are commonly used in this problem:

infinitesimal perturbations

$$Q(t) \to 0, \ \ t \to +\infty; \tag{3}$$

exponentially small perturbations

$$\|Q(t)\| \leq C(Q)\exp(-\sigma(Q)t), \quad C(Q) > 0, \ \ \sigma(Q) > 0; \tag{4}$$

σ-perturbations

$$\|Q(t)\| \leq C(Q)\exp(-\sigma t), \quad C(Q) > 0, \ \ \sigma > 0; \tag{5}$$

power perturbations

$$\|Q(t)\| \leq C(Q)t^{-\gamma}, \quad C(Q) > 0, \ \ \gamma > 0; \tag{6}$$

generalized power perturbations

$$\|Q(t)\| \leq C(Q)\exp(-\sigma\theta(t)), \quad C(Q) > 0, \ \ \sigma > 0, \tag{7}$$

$$\|Q(t)\| \leq C(Q)\exp(-\sigma(Q)\theta(t)), \quad C(Q) > 0, \ \ \sigma(Q) > 0, \tag{8}$$

where θ is a positive function satisfying some additional conditions;

infinitesimal average and integrable perturbations

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \|Q(t)\|\,dt = 0, \qquad \int_0^{+\infty} \|Q(t)\|\,dt < +\infty, \tag{9}$$

and their modifications with some positive weights φ and powers $p \geq 1$

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \varphi(\tau)\|Q(\tau)\|^p\,d\tau = 0; \qquad \int_0^{+\infty} \varphi(\tau)\|Q(t)\|^p\,dt < +\infty. \tag{10}$$

Generally to calculate $\Lambda(\mathfrak{M})$ we can construct an algorithm analogous to a famous Izobov algorithm for σ-exponent. Alternatively, in some cases (e.g., for classes (3), (4), (8)) we have formulas like the following Millionshcikov formula

$$\Omega(A) = \lim_{T\to+\infty} \overline{\lim_{k\to\infty}} \frac{1}{mT} \sum_{k=1}^{m} \ln \big\| X_A(kT, kT-T) \big\|, \tag{11}$$

for the central exponent. A smallness class $\mathfrak{M}$ is said to be a limit class if $\Lambda(\mathfrak{M})$ has a representation similar to (11). Any other general criteria for a class of perturbation to be a limit class are not known. In order to find such criteria or conditions (necessary or sufficient) we need to have a general definition of a smallness class.

Let $\mathbb{R}^{n\times n}$ be the normed vector space of all real $n \times n$-matrices with some norm $\|\cdot\|$ and $\mathrm{PC}_n(\mathbb{R}^+)$ be the space of bounded piecewise continuous functions $Q : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ where $\mathbb{R}^+ := [0, +\infty[$. The space $\mathrm{PC}_n(\mathbb{R}^+)$ with the operation of pointwise multiplication forms an algebra. We denote it by $\mathcal{K}_n$.

All commonly used classes (3)–(10) satisfy the following natural conditions.

(A0) $\mathfrak{M} \neq \varnothing$ and $\mathfrak{M} \neq \mathrm{PC}_n(\mathbb{R}^+)$.

(A1) The set $\mathfrak{M}$ is invariant with respect to Lyapunov transformations.

(A2) If $Q \in \mathfrak{M}$, $P \in \mathrm{PC}_n(\mathbb{R}^+)$, and $\|P(t)\| \le \|Q(t)\|$ for all $t \ge 0$, then $P \in \mathfrak{M}$.

The axioms (A0), (A1), and (A2) express the obvious necessary requirements to a smallness class. Unfortunately, these requirements are not sufficient. E.g., for an arbitrary nonempty $S \subset \mathbb{R}^+$ the set $I(S) := \{Q \in \mathrm{PC}_n(\mathbb{R}^+) : Q(t) = 0 \ \forall\, t \in S\}$ satisfies (A0), (A1), and (A2). It seems to be clear that $I(S)$ is not a smallness class in any sense. In order to eliminate these and some other improper classes, we propose two additional conditions.

(A3) There exists $Q \in \mathfrak{M}$ such that $Q(t) \neq 0$ for all $t \ge 0$.

(A4) $Q + P \in \mathfrak{M}$ for each $P, Q \in \mathfrak{M}$.

These conditions are valid for (3)–(10) too.

Definition 1. A set $\mathfrak{b} \subset \mathrm{PC}_1(\mathbb{R}^+)$, $\mathfrak{b} \neq \mathrm{PC}_1(\mathbb{R}^+)$, is said to be a one-dimensional smallness class if $\mathfrak{b}$ is an ideal of $\mathcal{K}_1$ and contains some strictly positive function.

Definition 2. A set $\mathfrak{M} \subset \mathrm{PC}_n(\mathbb{R}^+)$, $\mathfrak{M} \neq \mathrm{PC}_n(\mathbb{R}^+)$, is said to be a smallness class if $\mathfrak{M} = \mathfrak{b}^{n\times n}$ for some one-dimensional smallness class $\mathfrak{b}$.

Theorem. *Let $n \ge 2$. A set $\mathfrak{M} \subset \mathrm{PC}_n(\mathbb{R}^+)$ is a smallness class iff $\mathfrak{M}$ satisfies the axioms* (A0)–(A4).

References

[1] N. A. Izobov, Lyapunov exponents and stability. *Cambridge Scientific Publishers, Cambridge,* 2012.