Carathéodory Solutions to a Hyperbolic Differential Inequality with a Non-Positive Coefficient and Delayed Arguments

Alexander Lomtatidze

Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Brno, Czech Republic,
Institute of Mathematics, Academy of Sciences of the Czech Republic, branch in Brno, Brno, Czech Republic
E-mail: `lomtatidze@fme.vutbr.cz`

Jiří Šremr

Institute of Mathematics, Academy of Sciences of the Czech Republic, branch in Brno, Brno, Czech Republic
E-mail: `sremr@ipm.cz`

On the rectangle $\mathcal{D} = [a,b] \times [c,d]$ we consider the Darboux problem

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = p(t,x)u\big(\tau(t,x), \mu(t,x)\big) + q(t,x), \tag{1}$$

$$u(t,c) = \varphi(t) \quad \text{for } \ t \in [a,b], \quad u(a,x) = \psi(x) \quad \text{for } \ x \in [c,d], \tag{2}$$

where $p, q\colon \mathcal{D} \to \mathbb{R}$ are Lebesgue integrable functions, $\tau\colon \mathcal{D} \to [a,b]$ and $\mu\colon \mathcal{D} \to [c,d]$ are measurable functions, and $\varphi\colon [a,b] \to \mathbb{R}$, $\psi\colon [c,d] \to \mathbb{R}$ are absolutely continuous functions such that $\varphi(a) = \psi(c)$. By a *solution* to problem (1), (2) we mean a function $u\colon \mathcal{D} \to \mathbb{R}$ absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory[1] which satisfies equality (1) almost everywhere in $\mathcal{D}$ and verifies initial conditions (2).

We have introduced the following definition in [2].

Definition 1. Let $p\colon \mathcal{D} \to \mathbb{R}$ be a Lebesgue integrable function and $\tau\colon \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ be measurable functions. We say that a theorem on differential inequalities (maximum principle) holds for equation (1) and we write $(p,\tau,\mu) \in \mathcal{S}_{ac}(\mathcal{D})$ if for any function $u\colon \mathcal{D} \to \mathbb{R}$ absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory satisfying the inequalities

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} \geq p(t,x)u\big(\tau(t,x), \mu(t,x)\big) \quad \text{for a. e. } \ (t,x) \in \mathcal{D},$$

$$u(a,c) \geq 0, \quad \frac{\partial u(t,c)}{\partial t} \geq 0 \quad \text{for a. e. } \ t \in [a,b], \quad \frac{\partial u(a,x)}{\partial x} \geq 0 \quad \text{for a. e. } \ x \in [c,d],$$

the relation

$$u(t,x) \geq 0 \quad \text{for } \ (t,x) \in \mathcal{D} \tag{3}$$

holds.

It is also mentioned in [2] that under the assumption $(p,\tau,\mu) \in \mathcal{S}_{ac}(\mathcal{D})$, problem (1), (2) has a unique (Carathéodory) solution and this solution satisfies relation (3) provided $q(t,x) \geq 0$ for a. e. $(t,x) \in \mathcal{D}$, $\varphi(a) = \psi(c) \geq 0$ $\varphi'(t) \geq 0$ for a. e. $t \in [a,b]$, and $\psi'(x) \geq 0$ for a. e. $x \in [c,d]$. Moreover, some efficient conditions are given in [2] for the validity of the inclusion $(p,\tau,\mu) \in \mathcal{S}_{ac}(\mathcal{D})$ in the case, where $p(t,x) \geq 0$ for a. e. $(t,x) \in \mathcal{D}$. For the case where

$$p(t,x) \leq 0 \quad \text{for a. e. } \ (t,x) \in \mathcal{D}, \tag{4}$$

[1] This notion is introduced in [1].

we have presented a general sufficient condition in [2] guaranteeing the validity of the inclusion $(p,\tau,\mu)\in\mathcal{S}_{ac}(\mathcal{D})$ under the assumption that equation (1) is delayed in both arguments, i. e., if the inequalities

$$|p(t,x)|(\tau(t,x)-t)\le 0,\quad |p(t,x)|(\mu(t,x)-x)\le 0\quad\text{for a. e. }(t,x)\in\mathcal{D}\tag{5}$$

hold. Using that general result, we have also proved in [2] that if p, μ, and τ satisfy conditions (4) and (5), then $(p,\tau,\mu)\in\mathcal{S}_{ac}(\mathcal{D})$ provided

$$\iint\limits_{\mathcal{D}}|p(t,x)|\mathrm{d}t\mathrm{d}x\le 1.\tag{6}$$

Observe that assumption (5) is not restrictive in the case (4) because it is necessary as it is shown in [4]. Moreover, the number 1 on the right-hand side of inequality (6) is, in general, optimal (see [2, Example 6.2]). However, it does not mean that inequality (6) is necessary and cannot be weakened. Below, we give an efficient criteria for the validity of the inclusion $(p,\tau,\mu)\in\mathcal{S}_{ac}(\mathcal{D})$ in the case (4) which are optimal for equations “close” to the equation without argument deviations

$$\frac{\partial^2u(t,x)}{\partial t\,\partial x}=p(t,x)u(t,x)+q(t,x).$$

For this equation with a constant non-positive coefficient p the following proposition holds (see, e. g., [5, § 3.4] or [3, Example 8.1]).

Proposition 1. *Let $k\le 0$ and $p(t,x):=k$, $\tau(t,x):=t$, $\mu(t,x):=x$ for $(t,x)\in\mathcal{D}$. Then $(p,\tau,\mu)\in\mathcal{S}_{ac}(\mathcal{D})$ if and only if*

$$|k|\le\frac{j_0^2}{4(b-a)(d-c)},\tag{7}$$

where j_0 denotes the first positive zero of the Bessel function J_0.

Now for any $\nu>-1$, we denote by J_ν the Bessel function of the first kind and order ν and let j_ν be the first positive zero of the function J_ν. Moreover, we put

$$E_\nu(s):=\begin{cases}s^{-\nu}J_\nu(s) & \text{for }\ s>0,\\ 2^{-\nu}\dfrac{1}{\Gamma(1+\nu)} & \text{for }\ s=0,\end{cases}$$

where Γ is the gamma function of Euler.

Theorem 1. *Let $p\colon\mathcal{D}\to\mathbb{R}$ be a Lebesgue integrable function and $\tau\colon\mathcal{D}\to[a,b]$, $\mu\colon\mathcal{D}\to[c,d]$ be measurable functions satisfying conditions (4) and (5). Moreover, let there exist numbers $\lambda\in\,]0,1]$, $\alpha\in[0,1[\,$, and $\beta\in[0,\alpha]$ such that the inequalities*

$$\big[(t-a)(x-c)\big]^{1-\lambda}|p(t,x)|\le\frac{\lambda^2}{4}\,\frac{j_{-\alpha}^2}{\big[(b-a)(d-c)\big]^{\lambda}},\tag{8}$$

$$\begin{aligned}\big[(t-a)(x-c)\big]^{1-\lambda}\Big(E_{-\alpha}\big(z(\tau(t,x),x)\big)-E_{-\alpha}\big(z(t,x)\big)\Big)|p(t,x)|\le\\ \le\frac{\lambda^2\beta}{2}\,\frac{j_{-\alpha}^2}{\big[(b-a)(d-c)\big]^{\lambda}}\,E_{1-\alpha}\big(z(t,x)\big),\end{aligned}\tag{9}$$

$$\begin{aligned}\big[(t-a)(x-c)\big]^{1-\lambda}\Big(E_{-\alpha}\big(z(t,\mu(t,x))\big)-E_{-\alpha}\big(z(t,x)\big)\Big)|p(t,x)|\le\\ \le\frac{\lambda^2(\alpha-\beta)}{2}\,\frac{j_{-\alpha}^2}{\big[(b-a)(d-c)\big]^{\lambda}}\,E_{1-\alpha}\big(z(t,x)\big)\end{aligned}\tag{10}$$

are fulfilled a. e. on $\mathcal{D}$, where

$$z(t,x) := j_{-\alpha}\Big[\frac{(t-a)(x-c)}{(b-a)(d-c)}\Big]^{\frac{\lambda}{2}} \quad \textit{for } (t,x)\in\mathcal{D}.$$

Then a theorem on differential inequalities holds for equation (1)*, i. e.,* $(p,\tau,\mu)\in\mathcal{S}_{ac}(\mathcal{D})$*.*

Observe that if $\tau(t,x)=t$ for a. e. $(t,x)\in\mathcal{D}$, then the left-hand side of inequality (9) is equal to zero. Therefore, assumption (9) of Theorem 1 describes how "close" must $\tau(t,x)$ be to t and this "closeness" is understood through the composition of the functions $E_{-\alpha}$ and z. Similarly, "closeness" of $\mu(t,x)$ to x is required in assumption (10). A meaning of assumptions (9) and (10) of Theorem 1 is more transparent in the following corollary.

Corollary 1. *Let $p\colon\mathcal{D}\to\mathbb{R}$ be a Lebesgue integrable function and $\tau\colon\mathcal{D}\to[a,b]$, $\mu\colon\mathcal{D}\to[c,d]$ be measurable functions satisfying conditions* (4) *and* (5)*. Moreover, let there exist numbers $\alpha\in[0,1[$ and $\beta\in[0,\alpha]$ such that the inequalities*

$$|p(t,x)| \le \frac{j_{-\alpha}^2}{4(b-a)(d-c)}\,,$$
$$(x-c)\big(t-\tau(t,x)\big)|p(t,x)| \le \beta j_{-\alpha}^*, \quad (t-a)\big(x-\mu(t,x)\big)|p(t,x)| \le (\alpha-\beta) j_{-\alpha}^*$$

are fulfilled a. e. on $\mathcal{D}$, where

$$j_{-\alpha}^* := \frac{E_{1-\alpha}(j_{-\alpha})}{E_{1-\alpha}(0)}\,.$$

Then $(p,\tau,\mu)\in\mathcal{S}_{ac}(\mathcal{D})$*.*

References

[1] C. Carathéodory, Vorlesungen über reelle Funktionen. *B. G. Teubner, Leipzig und Berlin*, 1918.

[2] A. Lomtatidze, S. Mukhigulashvili, and J. Šremr, Nonnegative solutions of the characteristic initial value problem for linear partial functional-differential equations of hyperbolic type. *Math. Comput. Modelling* **47** (2008), No. 11-12, 1292–1313.

[3] J. Šremr, On the Darboux problem for linear hyperbolic functional-differential equations. *Institute of Mathematics, AS CR*, Preprint **IM-2013-8** (2013).

[4] J. Šremr, Some remarks on linear functional-differential inequalities of hyperbolic type. (Russian) *Ukrain. Mat. Zh.* **60** (2008), No. 2, 283–292; translation in *Ukrainian Math. J.* **60** (2008), No. 2, 327–337.

[5] F. G. Tricomi, Lezioni sulle equazioni a derivate parziali. (Italian) *Corso di analisi superiore, anno accademico* 1953–1954. *Editrice Gheroni, Torino*, 1954.