Properties of Stable and Attracting Sets of L-Systems

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Many of the current problems of predicting of physical and mechanical processes are reduced to study of solutions of evolution equations on large intervals of time in infinite-dimensional phase space. In recent decades, significant progress has been received in studying of such problems. This is connected to the fact that the attractors (compact attracting invariant sets) of many evolution equations generated by partial differential equations have been found. The phase space is not locally compact for most of evolution equations generated by partial differential equations, which prevents applying existing results of the theory of dynamical systems in locally compact spaces for studying of these equations, although the most profound results in studying of neighborhoods of attractors are obtained for dynamic systems in locally compact spaces. In [1] the class of dynamical systems which is called L-systems was introduced. These systems satisfy many properties of dynamical systems in locally compact spaces but their phase space is not necessarily locally compact.

Let $f(t,x)$ be a semidynamical system which is specified in a metric space X. Let Φ be the set of all the motions of semidynamical system $f(t,x)$.

Semidynamical system $f(t,x)$ is said to be L-system if it satisfies the following property: there exists a constant $\omega \geq 0$ such that if the sequence $\varphi_n(t) \in \Phi$ is bounded on any segment $[a-\omega; b]$, then there exists a subsequence of the sequence $\varphi_n(t)$ converging to some motion $\varphi(t) \in \Phi : \varphi_{n_k}(t) \to \varphi(t)$, $t \in [a;b]$.

Differential equations and inclusions, functional-differential inclusions, evolution parabolic equations generate L-systems in corresponding metric spaces.

We will use the following definitions and notations.

$D(x) = \{y \in X : \exists x_n \to x, \exists \varphi_n(t), \varphi_n(0) = x_n, \exists t_n \geq 0, \text{ such that } \varphi_n(t_n) \to y \text{ as } n \to \infty\}$;

$D(M) = \bigcup\limits_{x \in M} D(x)$;

The set M is weakly semiattracting, if $\exists \delta > 0$ such that for all $x \in S(M,\delta)$, for all motions $\varphi(t)$, $\varphi(0) = x$, there exists a sequence of the moments of time $t_n \to +\infty$ such that $\rho(\varphi(t_n), M) \to 0$ as $n \to \infty$;

$A_2(M) = \{x \in X : \forall \varphi_x, \rho(\varphi_x(t), M) \to 0 \text{ as } t \to +\infty\}$;

$A_\omega(M) = \{x \in X : \forall \varphi_x \exists t_n \to +\infty \text{ such that } \rho(\varphi_x(t_n), M) \to 0 \text{ as } n \to \infty\}$.

We say that a subset M of the space X satisfies condition A if there exist constants $\varepsilon > 0$, $L > 0$, $a > \omega$ such that for all $x \in S(M,\varepsilon)$, for all $\varphi_x(t)$ such that $\varphi_x(-T) \in S(M,\varepsilon)$ for some moment $T > a$, the inequality holds $\rho(\varphi_x(t), M) < L \ \forall t \in [-a; 0]$.

The following theorems are generalizations of well-known properties of semidynamical systems in locally compact metric spaces for L-systems [2].

Theorem 1. *Compact set M is stable if and only if $D(M) = M$.*

Theorem 2. *Let $f(t,x)$ be L-system, a compact set M be weakly semiattracting and satisfy the condition A. Then the set $D(M)$ is the least compact asymptotically stable set containing M and $A_\omega(M) = A_\omega(D(M)) = A(D(M))$.*

The set $B_0 \subseteq X$ is said to be an absorbing set of semidynamical system $f(t,x)$, if for each bounded set $B \subseteq X$ there exists $T \geq 0$ such that $f(t,B) \subseteq B_0$, $\forall t \geq T$.

In the following theorem which is a generalization of the well-known Yosidzava theorem ([2]) sufficient conditions of existence of the bounded absorbing set for L-system are given.

Theorem 3. *Let $f(t,x)$ be an L-system, and let there exist a continuous function $V(u), V : X \to R$, satisfying the following properties outside of some sphere $S(O, r_0)$, $O \in X$, $r_0 > 0$:*

(1) $V(u) \leq a(\rho(u,O))$, $\forall\, u \notin S(O,r_0)$, $a(r)$ – *positive continuous increasing function for* $r \geq 0$*;*

(2) $V(u) \to +\infty$ *as* $\rho(u,O) \to \infty$*;*

(3) *for every* $x \in X$ *and every* $\varphi_x(t) \notin S(O,r_0)$, $\forall\, t \in [t_1,t_2]$ *the inequality* $V(\varphi_x(t_1)) \geq V(\varphi_x(t_2))$ *holds;*

(4) *there does not exist a full motion* $\varphi_x^\infty(t)$ *such that* $V(\varphi_x^\infty(t)) = V(x) = const$ $\forall\, t \in R$ *and* $V(\varphi_x^\infty(R)) \cap S(0,r_0) = \varnothing$.

Then there exists a bounded absorbing set for L-system $f(t,x)$*.*

Nonempty set $U \subseteq X$ is said to be a global attractor of semidynamical system $f(t,x)$ if it satisfies the following properties:

(1) U is a compact;

(2) for each bounded set $B \subseteq X$, $\beta(f(t,B),U) \to 0$ as $t \to 0$;

(3) U is strict invariant, i. e. $f(U,t) = U$ $\forall\, t \geq 0$.

In the work [4] it is proved that if there exists a compact absorbing set $f(t,x)$, then there exists a global attractor $f(t,x)$. If $f(t,x)$ is L-system, then the last conditions could be weakened.

Theorem 4. *If there exists a bounded absorbing set for L-system* $f(t,x)$*, then there exists a global attractor for L-system* $f(t,x)$*.*

References

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