On the Properties of Regular Linear Differential Systems with Unbounded Coefficients

A. V. Konyukh

Belarusian State Economic University, Minsk, Belarus
E-mail: `al3128@gmail.com`

Consider the linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \geqslant 0, \tag{1}$$

where $n \geqslant 2$, with the piecewise continuous coefficient matrix. We denote the class of all such systems (1) by $\mathcal{M}_n^*$, and by $\mathcal{M}_n^0$ we denote its subclass, consisting of the systems, any nonzero solution of which has a finite Lyapunov exponent. Also we denote by $\mathcal{M}_n$ a subclass of $\mathcal{M}_n^*$, consisting of the systems with bounded on time semiaxis coefficient matrix. We will identify system (1) with its coefficient matrix and write thereby $A \in \mathcal{M}_n^*$. Lyapunov exponents of the system $A \in \mathcal{M}_n^0$ are denoted by $\lambda_1(A) \leqslant \dots \leqslant \lambda_n(A)$.

The class of the regular differential systems is defined by A. M. Lyapunov [1]. Let us formulate this definition as well as classical criteria of regularity, since we will need it in order to formulate Theorem 1. The system $A \in \mathcal{M}_n$ is called regular if and only if any of the following conditions holds:

(i) the equality $\lambda_1(A) + \dots + \lambda_n(A) = \varliminf\limits_{t\to+\infty} t^{-1} \int\limits_0^t \mathrm{Sp} A(\tau)\, d\tau$ is valid;

(ii) the Lyapunov exponents $\lambda_1(A) \leqslant \dots \leqslant \lambda_n(A)$ of the system A are symmetrical with respect to zero to the Lyapunov exponents $\lambda_1(-A^{\mathsf T}) \leqslant \dots \leqslant \lambda_n(-A^{\mathsf T})$ of the conjugate system, i.e., the equality $\lambda_k(A) = -\lambda_{n-k+1}(-A^{\mathsf T})$ holds for every $k = 1, \dots, n$;

(iii) there exists a generalized Lyapunov transformation reducing the system A to the diagonal system with constant coefficients;

(iv) for every normal basis $\{x_1(\cdot), \dots, x_n(\cdot)\}$ of the solutions to the system A their Lyapunov exponents are exact (i.e., for every $k \in \{1, \dots, n\}$ there exists $\lim\limits_{t\to\infty} t^{-1}\ln\|x_k(t)\|$), and for every $k = 1, \dots, n-1$ an angle $\gamma_k(t) = \angle(x_k(t), \mathrm{span}\{x_{k+1}(t), \dots, x_n(t)\})$, $t \geqslant 0$, has the exact and zero Lyapunov exponent (i.e., $\lim\limits_{t\to+\infty} t^{-1}\ln\gamma_k(t) = 0$, $k \in \{1, \dots, n-1\}$);

(v) there exists such fundamental system $\{x_1(\cdot), \dots, x_n(\cdot)\}$ of the solutions to the system A, the Lyapunov exponents of which are exact, and the angle $\angle(x_k(t), \mathrm{span}\{x_{k+1}(t), \dots, x_n(t)\})$, $t \geqslant 0$, has the exact and zero Lyapunov exponent for every $k = 1, \dots, n-1$.

The condition (i) represents the definition of the regular system [1] given by A. M. Lyapunov, and the conditions (ii) and (iii) are Perron [2] and Basov–Bogdanov–Grobman [3–5] criteria of regularity, respectively. The conditions (iv) and (v) give two different forms of Vinograd criterion [6] of regularity of a system.

We denote by $\mathcal{R}_n$ the class of the regular according to Lyapunov linear differential n-dimensional systems. In [3] the notion of regularity is extended to systems with unbounded on the semiaxis coefficients: a system is called regular [3] if it can be transformed to a diagonal system with constant coefficients by some generalized Lyapunov transformation (let us recall, that linear invertible for every $t \geqslant 0$ transformation $x = L(t)y$ is called generalized Lyapunov transformation if $\lim\limits_{t\to+\infty} L(t) = \lim\limits_{t\to+\infty} L^{-1}(t) = 0$). We denote by $\mathcal{R}_n^0$ the class of regular n-dimentional systems (in general, with unbounded coefficients). In particular, as it follows from the definition, the inclusion $\mathcal{R}_n^0 \subset \mathcal{M}_n^0$ holds.

The question naturally arises, which properties of the regular systems of $\mathcal{R}_n$ inherit systems of $\mathcal{R}_n^0$. It turns out that for the systems of $\mathcal{R}_n^0$ all properties (i)–(v) hold. Moreover, each of these properties can be taken as the definition of the regular system of class $\mathcal{R}_n^0$.

Theorem 1. *The system A belongs to the class $\mathcal{R}_n^0$ if and only if any of the conditions* (i)–(v) *is valid for it.*

However, Lyapunov criterion of the regularity of the system of $\mathcal{R}_n$ is well-known: the system is regular if and only if under some (and hence under any) Lyapunov transformation, reducing system to a triangular form, diagonal elements of the obtained coefficient matrix have exact integral mean values (moreover, the set of those mean values, taking into account their multiplicities, coincides with the set of the Lyapunov exponents of the system).

The above condition is necessary for the regularity of the system of $\mathcal{M}_n^0$ as well, as the following theorem shows.

Theorem 2. *For any reduction by generalized Lyapunov transformation* (*and, in particular, Lyapunov transformation*) *of the system $A \in \mathcal{R}_n^0$ to the triangular form, its diagonal coefficients have finite exact mean values, the set of which, taking into account their multiplicities, coincides with the set of the Lyapunov exponents of the system A.*

It appears that for the systems from the class $\mathcal{R}_n^0$ Lyapunov criterion of regularity, generally speaking, does not hold (i.e., the above condition is not sufficient – Theorem 2 is irreversible).

Theorem 3. *There exists an irregular system, Lyapunov exponents of any nonzero solutions of which are finite and exact, such that for any its reduction by generalized Lyapunov transformation* (*and, in particular, the Lyapunov transformation*) *to a triangular form, its diagonal coefficients have finite exact integral mean values, the set of which, taking into account their multiplicities, is the same for any such transformation.*

Nevertheless, as the following Theorem 4 shows, the basic property of the systems of $\mathcal{R}_n$ – to save conditional exponential stability, as well as the dimension of the exponentially stable manifold and asymptotic indicators of its solutions under perturbations of higher order of smallness – is also valid for the systems of $\mathcal{R}_n^0$.

We denote by $P_n^{(0)}$ a class of continuous vector-valued functions $f(\,\cdot\,,\,\cdot\,) : [0, +\infty) \times B_f \to \mathbb{R}^n$, where B_f is a closed ball in $\mathbb{R}^n$, centered at the origin (dependent on f; its radius is denoted by r_f), such that $f(t, 0) = 0$ for all $t \geqslant 0$ and $\|f(t, x_1) - f(t, x_2)\| \leqslant F(t)N(r)\|x_1 - x_2\|$ for all $x_1, x_2 \in B_f$ and $t \geqslant 0$, where $r = \max\{\|x_1\|, \|x_2\|\}$, and $F(\,\cdot\,)$ and $N(\,\cdot\,)$ are continuous (dependent on f) functions, defined respectively on $[0, +\infty)$ and $[0, r_f]$, satisfying the following conditions: Lyapunov exponent of the function $F(\,\cdot\,)$ is nonpositive and $N(r) = O(r^\varepsilon)$ as $r \to +0$ for some $\varepsilon > 0$ (dependent on f).

Theorem 4. *Let the system* (1) *belong to $\mathcal{R}_n^0$ and have exactly k distinct negative Lyapunov exponents: $\Lambda_1(A) < \cdots < \Lambda_k(A) < 0$ of multiplicities $n_1, \ldots, n_k$, respectively. Then for solutions of the system*

$$\dot{x} = A(t)x + f(t, x), \quad (t, x) \in [0, +\infty) \times B_f, \tag{2}$$

where $f \in P_n^{(0)}$, the following properties hold:

(1) *there exists a ball $B \subset \mathbb{R}^n$ and a sequence of enclosed manifolds $\mathbf{0} = M_0 \subset M_1 \subset \cdots \subset M_k \subset B$, such that $\dim M_i = n_i$, $i = 1, \ldots, k$, and every solution $x(\,\cdot\,)$ to the system* (2)*, beginning at $t = 0$ on $M_i \setminus M_{i-1}$, is extendable on $[0, +\infty)$ and for any $\delta > 0$ and for all $t \geqslant 0$ satisfies the two-sided estimate*

$$C_{\delta,x} \exp\bigl\{(\Lambda_i(A) - \delta)t\bigr\}\|x(0)\| \leqslant \|x(t)\| \leqslant C_\delta \exp\bigl\{(\Lambda_i(A) + \delta)t\bigr\}\|x(0)\|,$$

where the constant C_δ depends only on δ, and the constant $C_{\delta,x}$ – only on δ and on the selection of solution $x(\,\cdot\,)$;

(2) *any solution to the system* (2), *beginning at* $t = 0$ *in* $B \setminus M_k$, *leaves the ball* B *in finite time if* $\lambda_{n_1+\cdots+n_k+1}(A) > 0$, *and if* $\lambda_{n_1+\cdots+n_k+1}(A) = 0$, *then any such solution, continued inside* B *on* $[0, +\infty)$, *has zero exponent;*

(3) *for every* $t \geqslant 0$ *and* $i = 1, \dots, k$ *an orthogonal projection of manifold* $M_i(t) = \{x(t) \in \mathbb{R}^n : x(0) \in M_i\}$ *on lineal* $V_i(t)$, *formed at time* t *by the values of the solutions of* (1), *Lyapunov exponent of which does not exceed* $\Lambda_i(A)$, *is a continuous bijection onto its image, and the manifold* $M_i(t)$ *is tangent to lineal* $V_i(t)$ *at the origin;*

(4) *if the vector function* $f(\cdot, \cdot)$ *is continuously differentiable in* x, *then for every* $t \geqslant 0$ *and* $i = 1, \dots, k$ *the manifold* $M_i(t)$ *belongs to the class* C^1.

References

[1] A. M. Lyapunov, Sobr. soch. V 6-ti t. T. 2. *Izd-vo AN SSSR, M.-L.*, 1956.

[2] O. Perron, Die Ordnungszahlen linearer Differentialgleichungssysteme. (German) *Math. Z.* **31** (1930), No. 1, 748–766.

[3] V. P. Basov, On a structure of a solution to a regular system. *Vestn. Leningr. Un-ta. Ser. matem., fiz. i chim.*, 1952, No. 12, 3–8.

[4] D. M. Grobman, Characteristic exponents of systems near to linear ones. (Russian) *Mat. Sbornik N.S.* **30(72)**, (1952), 121–166.

[5] Yu. S. Bogdanov, On the theory of systems of linear differential equations. (Russian) *Dokl. Akad. Nauk SSSR (N.S.)* **104** (1955), 813–814.

[6] E. A. Barabanov and A. V. Konyukh, On a class of regular linear differential systems with unbounded coefficients. (Russian) *Differ. Uravn.* **46** (2010), No. 12, 1675–1692; translation in *Differ. Equ.* **46** (2010), No. 12, 1677–1693.