The Existence of o-Solutions of Quasi-Linear Two-Dimensional System of Differential Equations in the Case when the Roots of the Characteristic Equation are $0 \neq \lambda_1(+\infty) \in \mathbb{R},\ \lambda_2(+\infty) = 0$

Liliya Koltsova and Alexander Kostin

I. I. Mechnikov Odessa National University, Odessa, Ukraine

E-mail: a_kostin@ukr.net, koltsova.liliya@gmail.com

We study the asymptotic for $t \to +\infty$ of o-solutions of real system of differential equations:

$$\begin{cases} y_1' = \alpha_1(t) f_1(t, y_1, y_2), \\ y_2' = \alpha_2(t) f_2(t, y_1, y_2), \end{cases} \qquad (t, y_1, y_2) \in G, \tag{1}$$

$G = \{t \in \Delta = \Delta(t_0) = [t_0, +\infty) \subset \mathbb{R};\ |y_k| \leqslant b,\ b \in (0, +\infty)\ (k = 1, 2)\}$, $0 < \alpha_k(t) \in C(\Delta)$, $\int\limits_{t_0}^{+\infty} \alpha_k(t)\,dt = +\infty$ $(k = 1, 2)$; $\exists \lim\limits_{t \to +\infty} \frac{\alpha_2(t)}{\alpha_1(t)} \in [0, +\infty)$; $f_k(t, y_1, y_2) \in C^{1\,2\,2}_{t\,y_1 y_2}(G)$, $\exists\, f_k(+\infty, 0, 0) = 0$, $\exists\, \frac{\partial^{i+j} f_k}{\partial y_1^i y_2^j}(+\infty, 0, 0) \in \mathbb{R}$ $(k = 1, 2;\ i, j \in \{0, 1, 2\};\ (i + j) \in \{1, 2\})$; $0 \neq \lambda_k(t) \in C^1_\Delta$ $(k = 1, 2)$ – the real roots of the characteristic equation

$$\begin{bmatrix} \dfrac{\partial f_1(t, 0, 0)}{\partial y_1} - \lambda & \dfrac{\partial f_1(t, 0, 0)}{\partial y_2} \\ \dfrac{\alpha_2(t)}{\alpha_1(t)} \dfrac{\partial f_2(t, 0, 0)}{\partial y_1} & \dfrac{\alpha_2(t)}{\alpha_1(t)} \dfrac{\partial f_2(t, 0, 0)}{\partial y_2} - \lambda \end{bmatrix} = 0,$$

where $\lambda_1(t) \to \lambda_1^0 \in \mathbb{R}$, $\lambda_1^0 \neq 0$, $\lambda_2(t) = o(1)$ for $t \to +\infty$, $(\frac{\partial f_1(+\infty,0,0)}{\partial y_2})^2 + (\lambda_1^0 - \frac{\partial f_1(+\infty,0,0)}{\partial y_1})^2 > 0$.

The case $\lambda_k(+\infty) \neq 0$ $(k = 1, 2)$ was discussed in an article [1] for systems of dimension $n \geq 1$. The results obtained in the article [1] are not applicable for system (1) with predetermined conditions.

We introduce the following notation.

$$p_{ij}(t) = \frac{\partial f_i(t, 0, 0)}{\partial y_j} \ (i, j = 1, 2);$$

$$q_1(t) = \frac{f_1(t, 0, 0) p_{12}(t) + \frac{\alpha_2(t)}{\alpha_1(t)} f_2(t, 0, 0)(\lambda_1(t) - p_{11}(t))}{\sqrt{p_{12}^2(t) + (\lambda_1(t) - p_{11}(t))^2}},$$

$$q_2(t) = \frac{-f_1(t, 0, 0)(\lambda_1(t) - p_{11}(t)) + \frac{\alpha_2(t)}{\alpha_1(t)} f_2(t, 0, 0) p_{12}(t)}{\sqrt{p_{12}^2(t) + (\lambda_1(t) - p_{11}(t))^2}}.$$

The functions $p_{ij}(t)$, $\tilde{q}_i(t) \in C^1_\Delta$, $p_{ij}(+\infty) = p_{ij}^0$, $\tilde{q}_i(+\infty) = 0$ $(i, j = 1, 2)$ in view of the conditions of system (1).

We formulate some theorems.

Theorem 1. *Let the system* (1) *meet the additional conditions:*

(1) $\frac{\lambda_2'(t)}{\alpha_1(t)\lambda_2^2(t)} \to \ell_{11} \in \mathbb{R}$ *for* $t \to +\infty$, $\int\limits_{t_0}^{+\infty} \alpha_1(t)\lambda_2(t)dt = \infty$;

(2) $q_1(t)=o(\frac{q_2(t)}{\lambda_2(t)})$, $\frac{q_2(t)}{\lambda_2^2(t)}\to\ell_{12}\in\mathbb{R}$, $\frac{q_2'(t)}{\alpha_1(t)\lambda_2(t)\tilde{q}_2(t)}\to\ell_{13}\in\mathbb{R}$ *for* $t\to+\infty$;

(3) $|p_{12}'(t)|+|\lambda_1'(t)-p_{11}'(t)|=o(\alpha_1(t)\lambda_2(t))$ *for* $t\to+\infty$.

Then in $\Delta(t_1)\subset\Delta$ *the system* (1) *for* $t\to+\infty$ *exists at least one o-solution* $y_k\in C^1_{\Delta(t_1)}$ $(k=1,2)$ *of the form:*

$$y_1(t)\,\dot\sim\,\frac{q_2(t)}{\lambda_2(t)}\,,\quad y_2(t)\,\dot\sim\,\frac{q_2(t)}{\lambda_2(t)}\,.^* \tag{2}$$

Theorem 2. *Let the system* (1) *meet the additional conditions:*

(1) $\frac{\lambda_2'(t)}{\alpha_1(t)\lambda_2^2(t)}\to\ell_{21}\in\mathbb{R}$ *for* $t\to+\infty$, $\int\limits_{t_0}^{+\infty}\alpha_1(t)\lambda_2(t)dt=\infty$;

(2) $q_1(t)=o(\lambda_2(t))$, $\frac{q_2(t)}{\lambda_2^2(t)}\to\ell_{22}\in\mathbb{R}$ *for* $t\to+\infty$;

(3) $|p_{12}'(t)|+|\lambda_1'(t)-p_{11}'(t)|=o(\alpha_1(t)\lambda_2(t))$ *for* $t\to+\infty$.

Then in $\Delta(t_1)\subset\Delta$ *the system* (1) *for* $t\to+\infty$ *exists at least one o-solution* $y_k\in C^1_{\Delta(t_1)}$ $(k=1,2)$ *of the form:*

$$y_1(t)\,\dot\sim\,\lambda_2(t),\quad y_2(t)\,\dot\sim\,\lambda_2(t). \tag{3}$$

Theorem 3. *Let the system* (1) *meet the additional conditions:*

(1) $\alpha_1(t)\lambda_2(t)\to\ell_{31}\in\mathbb{R}$, $\ell_{31}\neq 0$ *for* $t\to+\infty$;

(2) $\frac{q_2(t)}{\lambda_2(t)}=o(q_1(t))$, $\alpha_1(t)q_1(t)=o(1)$, $q_1^2(t)=o(q_2(t))$ *for* $t\to+\infty$;

(3) $\frac{q_1'(t)}{\alpha_1(t)q_1(t)}\to\ell_{32}\in\mathbb{R}$, $\ell_{32}\neq\lambda_1^0$, $(\frac{q_2(t)}{\lambda_2(t)})'=o(\frac{q_2(t)}{\lambda_2(t)})$ *for* $t\to+\infty$;

(4) $|p_{12}'(t)|+|\lambda_1'(t)-p_{11}'(t)|=o(1)$ *for* $t\to+\infty$.

Then in $\Delta(t_1)\subset\Delta$ *the system* (1) *for* $t\to+\infty$ *exists at least one o-solution* $y_k\in C^1_{\Delta(t_1)}$ $(k=1,2)$ *of the form:*

$$y_1(t)\,\dot\sim\,q_1(t),\quad y_2(t)\,\dot\sim\,\frac{q_2(t)}{\lambda_2(t)}. \tag{4}$$

Theorem 4. *Let the system* (1) *meet the additional conditions:*

(1) $\alpha_1(t)\lambda_2(t)\to\ell_{41}\in\mathbb{R}$, $\ell_{41}\neq 0$, $\lambda_2(t)=o(q_1(t))$, $\frac{\alpha_1(t)q_2(t)}{\lambda_2(t)}\to\ell_{42}\in\mathbb{R}$ *for* $t\to+\infty$;

(2) $\lambda_2'(t)=o(\lambda_2(t))$, $\frac{q_1'(t)}{\alpha_1(t)q_1(t)}\to\ell_{43}\in\mathbb{R}$, $\ell_{43}\neq\lambda_1^0$ *for* $t\to+\infty$;

(3) $|p_{12}'(t)|+|\lambda_1'(t)-p_{11}'(t)|=o(1)$*for* $t\to+\infty$.

Then in $\Delta(t_1)\subset\Delta$ *the system* (1) *for* $t\to+\infty$ *exists at least one o-solution* $y_k\in C^1_{\Delta(t_1)}$ $(k=1,2)$ *of the form:*

$$y_1(t)\,\dot\sim\,q_1(t),\quad y_2(t)\,\dot\sim\,\lambda_2(t). \tag{5}$$

Theorem 5. *Let the system* (1) *meet the additional conditions:*

(1) $\frac{q_2(t)}{\lambda_2(t)}=o(q_1(t))$, $q_1^2(t)=o(\frac{q_2(t)}{\lambda_2(t)})$ *for* $t\to+\infty$;

$^*f_i\,\dot\sim\,f_j$ $(i\neq j)$ means that $\exists\lim\limits_{t\to+\infty}\frac{f_i}{f_j}\neq 0,\pm\infty$.

(2) $\frac{q_1'(t)}{\alpha_1(t)q_1(t)} \to \ell_{51} \in \mathbb{R}$, $\ell_{51} \neq \lambda_1^0$, $[(\frac{q_2(t)}{\lambda_2(t)})' \frac{\lambda_2(t)}{\alpha_1(t)q_2(t)}] \to \ell_{52} \in \mathbb{R}$, $\ell_{52} \neq 0$ *for* $t \to +\infty$;

(3) $|p_{12}'(t)| + |\lambda_1'(t) - p_{11}'(t)| = o(\alpha_1(t))$ *for* $t \to +\infty$.

Then in $\Delta(t_1) \subset \Delta$ *the system* (1) *for* $t \to +\infty$ *exists at least one o-solution* $y_k \in \mathrm{C}^1_{\Delta(t_1)}$ $(k = 1, 2)$ *of the form* (4).

Theorem 6. *Let the system* (1) *meet the additional conditions:*

(1) $\lambda_2(t) = o(q_1(t))$, $q_1^2(t) = o(\lambda_2(t))$, $\frac{q_2(t)}{\lambda_2(t)} \to \ell_{61} \in \mathbb{R}$ *for* $t \to +\infty$;

(2) $\frac{q_1'(t)}{\alpha_1(t)q_1(t)} \to \ell_{62} \in \mathbb{R}$, $\ell_{62} \neq \lambda_1^0$, $\frac{\lambda_2'(t)}{\alpha_1(t)\lambda_2(t)} \to \ell_{63} \in \mathbb{R}$, $\ell_{63} \neq 0$ *for* $t \to +\infty$;

(3) $|p_{12}'(t)| + |\lambda_1'(t) - p_{11}'(t)| = o(\alpha_1(t))$ *for* $t \to +\infty$.

Then in $\Delta(t_1) \subset \Delta$ *the system* (1) *for* $t \to +\infty$ *exists at least one o-solution* $y_k \in \mathrm{C}^1_{\Delta(t_1)}$ $(k = 1, 2)$ *of the form* (5).

The constants $\ell_{ij} \in \mathbb{R}$ $(i = \overline{1,6},\ j \in \{1, 2, 3\})$ take part in finding the exact asymptotic of o-solutions.

References

[1] A. V. Kostin, Asymptotics of the regular solutions of nonlinear ordinary differential equations. (Russian) *Differ. Uravn.* **23** (1987), No. 3, 524–526.

[2] V. A. Kostin and S. V. Pisareva, Evolution equations with singularities in generalized Stepanov spaces. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* **2007**, No. 6, 35–44; translation in *Russian Math. (Iz. VUZ)* **51** (2007), No. 6, 32–41.