Asymptotic Property and Semi-Discrete Scheme for One System of Nonlinear Partial Integro-Differential Equations

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Based on Maxwell's system [1] the following kind of nonlinear integro-differential model arises for mathematical modeling of the process of penetrating of magnetic field in the substance [2]

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\left(\int_0^t |\operatorname{rot} H|^2\, d\tau\right)\operatorname{rot} H\right], \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that the system of the integro-differential equations (1) is complex. Equations and systems of type (1) still yield to the investigation for special cases (see, for example, [2]–[11] and references therein).

If the magnetic field has the form $H = (0, U, V)$ and $U = U(x,t)$, $V = V(x,t)$, then we have

$$\operatorname{rot}(a(S)\operatorname{rot} H) = \left(0, -\frac{\partial}{\partial x}\left(a(S)\frac{\partial U}{\partial x}\right), -\frac{\partial}{\partial x}\left(a(S)\frac{\partial V}{\partial x}\right)\right),$$

and from (1) we obtain the following system of nonlinear integro-differential equations

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial V}{\partial x}\right], \tag{2}$$

where

$$S(x,t) = \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau. \tag{3}$$

In [6] the asymptotic behavior of solutions as $t \to \infty$ of initial-boundary value problem for system (2), (3) with the homogeneous boundary conditions in the norm of the space H^1 is given. Here and below we use usual Sobolev spaces $H^k(0,1)$.

In [8] some generalization of the system of type (2), (3) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time, but independent of the space coordinates, the process of penetration of the magnetic field into the material is modeled by following averaged integro-differential model:

$$\frac{\partial U}{\partial t} = a(S)\frac{\partial^2 U}{\partial x^2}, \quad \frac{\partial V}{\partial t} = a(S)\frac{\partial^2 V}{\partial x^2}, \tag{4}$$

where

$$S(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\, d\tau. \tag{5}$$

The existence and uniqueness of solutions of the initial-boundary value problems for the models of type (2), (3) and (4), (5) are studied in many works (see, for example, [2]–[6], [8] and reference therein).

Our aim is to study the asymptotic behavior of solutions as $t \to \infty$ and semi-discrete scheme for the initial-boundary value problem for system (4), (5) for the case $a(S) = (1+S)^p$, $0 < p \le 1$.

In the domain $[0,1] \times [0,\infty)$ for the system (4), (5) we consider the following initial-boundary value problem

$$\begin{gathered} U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, \quad t \ge 0, \\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad x \in [0,1], \end{gathered} \tag{6}$$

where U_0 and V_0 are given functions.

The following statement is valid.

Theorem 1. *If $a(S) = (1+S)^p$, $0 < p \le 1$, U_0, $V_0 \in H^3(0,1) \cap H_0^1(0,1)$, then for the solution of problem* (4)–(6) *the following asymptotic relations hold as $t \to \infty$:*

$$\begin{gathered} \left|\frac{\partial U(x,t)}{\partial x}\right| \le C \exp\left(-\frac{t}{2}\right), \quad \left|\frac{\partial V(x,t)}{\partial x}\right| \le C \exp\left(-\frac{t}{2}\right), \\ \left|\frac{\partial U(x,t)}{\partial t}\right| \le C \exp\left(-\frac{t}{2}\right), \quad \left|\frac{\partial V(x,t)}{\partial t}\right| \le C \exp\left(-\frac{t}{2}\right). \end{gathered}$$

Here and below C denotes a positive constant.

Now let us consider the semi-discrete scheme for (4)–(6) problem. On [0,1] let us introduce a net with mesh points denoted by $x_i = ih$, $i = 0, 1, \ldots, M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. The semi-discrete approximation at (x_i, t) are designed by $u_i = u_i(t)$ and $v_i = v_i(t)$. The exact solution to the problem at (x_i, t) is denoted by $U_i = U_i(t)$ and $V_i = V_i(t)$. At points $i = 1, 2, \ldots, M-1$, the integro-differential equation will be replaced by approximation of the space derivatives by a forward and backward differences.

Let us correspond to problem (4)–(6) the following semi-discrete scheme:

$$\begin{gathered} \frac{du_i}{dt} = \left(1 + h\int_0^t \sum_{k=1}^M \left[(u_{\overline{x},k})^2 + (v_{\overline{x},k})^2\right] d\tau\right)^p u_{\overline{x}x,\,i}, \\ \frac{dv_i}{dt} = \left(1 + h\int_0^t \sum_{k=1}^M \left[(u_{\overline{x},k})^2 + (v_{\overline{x},k})^2\right] d\tau\right)^p v_{\overline{x}x,\,i}, \\ i = 1, 2, \ldots, M-1, \end{gathered} \tag{7}$$

$$u_0(t) = u_M(t) = v_0(t) = v_M(t) = 0, \tag{8}$$

$$u_i(0) = U_{0,i}, \quad v_i(0) = V_{0,i}, \quad i = 0, 1, \ldots, M. \tag{9}$$

So, we obtained the Cauchy problem (7)–(9) for nonlinear system of ordinary integro-differential equations.

It is not difficult to obtain the following estimates:

$$\|u(t)\|^2 + \int_0^t \|u_{\overline{x}}]|^2 \, d\tau \le C, \quad \|v(t)\|^2 + \int_0^t \|v_{\overline{x}}]|^2 \, d\tau \le C, \tag{10}$$

where

$$\|w(t)\|^2 = \sum_{i=1}^{M-1} w_i^2(t) h, \quad \|w_{\overline{x}}]|^2 = \sum_{i=1}^{M} w_{\overline{x},\,i}^2(t) h.$$

The a priori estimates (10) guarantee the global solvability of the problem (7)–(9).

The following statement is true.

Theorem 2. *If* $a(S) = (1+S)^p$, $0 < p \le 1$, *and problem* (4)–(6) *has a sufficiently smooth solution* $U(x,t)$, $V(x,t)$, *then the solution of problem* (7)-(9) $u = u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t))$, $v = v(t) = (v_1(t), v_2(t), \dots, v_{M-1}(t))$ *tends to* $U = U(t) = (U_1(t), U_2(t), \dots, U_{M-1}(t))$, $V = V(t) = (V_1(t), V_2(t), \dots, V_{M-1}(t))$ *as* $h \to 0$ *and the following estimates are true*

$$\|u(t) - U(t)\| \le Ch, \quad \|v(t) - V(t)\| \le Ch.$$

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