Initial Value Problems for Nonlinear Singular Hyperbolic Equations of Higher Order with Two Independent Variables

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For the hyperbolic partial differential equation

$$u^{(m,n)} = f(x, y, u) \tag{1}$$

the Darboux problem

$$\begin{aligned} u^{(i,0)}(x, 0) = 0 \quad &\text{for} \;\; 0 \le x \le a \;\; (i = 0, \dots, m-1), \\ u^{(0,k)}(\gamma(y), y) = 0 \quad &\text{for} \;\; 0 \le y \le b \;\; (k = 0, \dots, n-1), \end{aligned} \tag{2}$$

and the Cauchy problem

$$u^{(i,n)}(x, \varphi(x)) = 0, \quad u^{(0,k)}(x, \varphi(x)) = 0 \quad \text{for} \;\; 0 \le x \le a \;\; (i = 0, \dots, m-1; \;\; k = 0, \dots, n-1) \tag{3}$$

are studied. Here m and n are positive integers,

$$u^{(i,k)}(x, y) = \frac{\partial^{i+k} u(x, y)}{\partial x^i \partial y^k} \;\; (i = 0, \dots, m; \;\; k = 0, \dots, n),$$

and $\gamma : [0, b] \to [0, a]$ and $\varphi : [0, a] \to [0, b]$ $(i, k = 1, 2)$ are continuously differentiable functions such that

$$\begin{gathered} \gamma(0) = 0, \;\; \gamma(y) < a, \;\; \gamma'(y) \ge 0 \quad \text{for} \;\; 0 < y < b, \\ \varphi(0) = b, \;\; \varphi'(x) < 0 \quad \text{for} \;\; 0 \le x \le a. \end{gathered}$$

Problem (1), (2) (problem (1), (3)) is considered in the domain $D_0 = \{(x, y) \in \mathbb{R}^2 : \; \gamma(y) < x < a, \; 0 < y < b\}$ (in the domain $D = \{(x, y) \in \mathbb{R}^2 : \; 0 < x < a, \; \varphi(x) < y < b\}$). By $\overline{D}_0$ and $\overline{D}$ we denote the closure of the domains D_0 and D.

The existence theorems formulated below concern the case where f is a nonnegative function defined in the domain $D_0 \times (0, +\infty)$ (in the domain $D \times (0, +\infty)$) and satisfies the inequality

$$p(x, y) z^{-\lambda} \le f(x, y, z) \le q(x, y) z^{-\lambda} + r(x, y)(1 + z), \tag{4}$$

where p, q and r are nonnegative Lebesgue integrable functions in D_0 (in D).

Furthermore, $f(\,\cdot\,, \,\cdot\,, z) : D_0 \to [0, +\infty)$ $(f(\,\cdot\,, \,\cdot\,, z) : D \to [0, +\infty))$ is measurable for every $z \in (0, +\infty)$, and $f(x, y, \,\cdot\,) : (0, +\infty) \to [0, +\infty)$ is continuous almost for every $(x, y) \in D_0$ (almost for every $(x, y) \in D$).

Introduce the functions

$$\mathcal{P}_0(x, y) = \int\limits_0^y (y - t)^{n-1} \bigg(\int\limits_{\gamma(y)}^x (x - s)^{m-1} p(s, t) \, ds \bigg) \, dt \quad \text{for} \;\; (x, y) \in D_0$$

and

$$\mathcal{P}(x,y)=\int\limits_{\varphi(x)}^{y}(y-t)^{n-1}\Bigg(\int\limits_{\psi(t)}^{x}(x-s)^{m-1}p(s,t)\,ds\Bigg)\,dt \quad \text{for } (x,y)\in D,$$

where ψ is the function inverse to φ.

If

$$\mathcal{P}_0(x,y)>0 \quad \text{for } (x,y)\in D_0, \tag{5}$$

then by conditions (4), there exists a set $D_0'\subset D_0$ of a positive measure such that

$$\lim_{z\to 0+} f(x,y,z)=+\infty \quad \text{for } (x,y)\in D_0'.$$

Similarly, if

$$\mathcal{P}_0(x,y)>0 \quad \text{for } (x,y)\in D, \tag{6}$$

then there exists a set $D'\subset D$ of a positive measure such that

$$\lim_{z\to 0+} f(x,y,z)=+\infty \quad \text{for } (x,y)\in D'.$$

Consequently equation (1) has singularity with resect to the phase variable. For such case problems (1), (2) and (1), (3) have not been studied before.

By a solution of problem (1), (2) we understand a continuous function $u:\overline{D}_0\to[0,+\infty)$ which is positive in the domain D_0 and satisfies in that domain the integral equation

$$u(x,y)=\frac{1}{(m-1)!(n-1)!}\int\limits_{\gamma(y)}^{x}(x-s)^{m-1}\Bigg(\int\limits_{0}^{y}(y-t)^{n-1}f(s,t,u(s,t))\,ds\Bigg)\,dt.$$

By a solution of problem (1), (3) we understand a continuous function $u:\overline{D}\to[0,+\infty)$ which is positive in the domain D and satisfies in that domain the integral equation

$$u(x,y)=\frac{1}{(m-1)!(n-1)!}\int\limits_{\varphi(x)}^{y}(y-t)^{n-1}\Bigg(\int\limits_{\psi(t)}^{x}(x-s)^{m-1}f(s,t,u(s,t))\,ds\Bigg)\,dt.$$

Theorem 1. *If along with* (4) *and* (5) *the condition*

$$\iint\limits_{D_0} q(x,y)(\mathcal{P}_0(x,y))^{-\frac{\lambda}{\lambda+1}}\,dx\,dy<+\infty$$

holds, then problem (1), (2) *has at least one solution.*

Corollary 1. *Let the condition*

$$l_1(x-\gamma(y))^{\alpha}y^{\beta}z^{-\lambda}\le f(x,y,z)\le l_2(x-\gamma(y))^{\alpha}y^{\beta}z^{-\lambda}+r(x,y)(1+z)$$

hold in $D_0\times(0,+\infty)$, *where* $l_1>0$, $l_2>0$, $\lambda>0$, α *and* β *are some constants. Then problem* (1), (2) *is solvable if and only if*

$$\alpha>(m-1)\lambda-1,\quad \beta>(n-1)\lambda-1.$$

Remark. If $\gamma(y)\equiv 0$, then problem (1), (2) is a characteristic value problem. Thus Theorem 1 and Corollary 1 contain optimal sufficient conditions of solvability of a characteristic value problem.

Theorem 2. *If along with* (4) *and* (6) *the condition*

$$\iint\limits_D q(x,y)(\mathcal{P}(x,y))^{-\frac{\lambda}{\lambda+1}}\, dx\, dy < +\infty$$

holds, then problem (1), (3) *has at least one solution.*

Corollary 2. *Let the condition*

$$l_1(y-\varphi(x))^{\mu} z^{-\lambda} \le f(x,y,z) \le l_2(y-\varphi(x))^{\mu} z^{-\lambda} + r(x,y)(1+z)$$

hold in $D \times (0,+\infty)$, *where* $l_1 > 0$, $l_2 > 0$, $\lambda > 0$ *and* μ *are some constants. Then problem* (1), (3) *is solvable if and only if*

$$\mu > (m+n-1)\lambda - 1.$$