Positive Solutions of Nonlinear Nonlocal Problems for Second Order Singular Differential Equations

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In a finite interval $]a,b[$ we consider the nonlinear differential equation

$$u'' = f(t,u) \tag{1}$$

with the nonlinear nonlocal boundary conditions of one of the following two types:

$$u(a) = \ell_1(u), \quad u(b) = \ell_2(u) \tag{2}$$

and

$$u(a) = \ell_1(u), \quad u'(b) = \ell_2(u). \tag{3}$$

Here, $f :]a,b[\times]0,+\infty[\to\mathbb{R}_+$ is a measurable in the first and continuous in the second argument function, $\mathbb{R}_+ = [0,+\infty[$, and $\ell_i : C([a,b];\mathbb{R}_+) \to \mathbb{R}_+$ $(i=1,2)$ are continuous functionals.

Let $C([a,b];\mathbb{R})$ be a space of continuous functions $u : [a,b] \to \mathbb{R}$ with the norm

$$\|u\| = \max\big\{|u(t)| : \ a \le t \le b\big\};$$

$C([a,b];\mathbb{R}_+) = \big\{u \in C([a,b];\mathbb{R}) : \ u(t) \ge 0 \text{ for } a \le t \le b\big\}$, and let $\widetilde{C}^1_{loc}(]a,b[\,;\mathbb{R})$ be a space of continuously differentiable functions $u :]a,b[\to\mathbb{R}$ whose first derivative is absolutely continuous on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.

A function $u \in C([a,b];\mathbb{R}) \cap \widetilde{C}^1_{loc}(]a,b[\,;\mathbb{R})$ is said to be a **positive solution** of the equation (1) if

$$u(t) > 0 \ \text{ for } \ a < t < b$$

and

$$u''(t) = f(t,u(t)) \ \text{ for almost all } \ t \in]a,b[\,.$$

A positive solution u of the equation (1) is said to be a **positive solution of the problem** (1), (2) (**of the problem** (1), (3)) if it satisfies the equalities (2) (has a finite limit $u'(b) = \lim\limits_{t\to b} u'(t)$ and satisfies the equalities (3)).

The theorems below on the existence of a positive solution of the problem (1), (2) (of the problem (1), (3)) deal with the cases where the function f in the domain $]a,b[\times]0,+\infty[$ satisfies the inequality

$$p_0(t) \le -q(x)f(t,x) \le p_1(t) + p_2(t)(1+x), \tag{4}$$

and the functionals ℓ_i $(i=1,2)$ on the set $C([a,b];\mathbb{R}_+)$ satisfy one of the following three conditions:

$$\ell_i(u) \le r\|u\| + r_0 \ \ (i=1,2), \tag{5}$$

$$\ell_1(u) \le r\|u\| + r_0, \quad \ell_2(u) \le \|u\|_{[a,t_0]}, \tag{6}$$

and

$$\ell_1(u) + (b-a)\ell_2(u) \le r\|u\| + r_0, \tag{7}$$

where $p_i :]a,b[\to\mathbb{R}_+$ $(i=0,1,2)$ are measurable functions, $q :]0,+\infty[\to]0,+\infty[$ is a continuous, nondecreasing function, r and r_0 are nonnegative constants, $t_0 \in]a,b[$ and

$$\|u\|_{[a,t_0]} = \max\big\{u(t) : \ a \le t \le t_0\big\}.$$

We are, mainly, interested in the case

$$\lim_{x\to 0} q(x) = 0 \ \text{ and } \ p_0(t) > 0 \ \text{ for } \ t \in I,$$

where $I \subset [a,b]$ is a set of positive measure. In this case,

$$\lim_{x\to 0} f(t,x) = -\infty \ \text{ for } \ t \in I,$$

i.e., the equation (1) is singular with respect to the phase variable.

Let

$$q(+\infty) = \lim_{x\to +\infty} q(x).$$

The following theorems are valid.

Theorem 1. *If along with* (4) *and* (5) *the conditions*

$$0 < \int_a^b (t-a)(b-t)p_i(t)\,dt < +\infty \ \ (i=0,1), \tag{8}$$

$$r < 1, \quad \int_a^b (t-a)(b-t)p_2(t)\,dt < (1-r)(b-a)q(+\infty)$$

are fulfilled, then the problem (1), (2) *has at least one positive solution.*

Theorem 2. *If along with* (4), (6) *and* (8) *the conditions*

$$r < 1, \quad \int_a^b (t-a)(b-t)p_2(t)\,dt < (1-r)(b-t_0)q(+\infty)$$

are fulfilled, then the problem (1), (2) *has at least one positive solution.*

Theorem 3. *If along with* (4) *and* (7) *the conditions*

$$0 < \int_a^b (t-a)p_i(t)\,dt < +\infty \ \ (i=0,1),$$

$$r < 1, \quad \int_a^b (t-a)p_2(t)\,dt < (1-r)(b-a)q(+\infty)$$

are fulfilled, then the problem (1), (3) *has at least one positive solution.*

Note that if the conditions of Theorem 1 or 2 (of Theorem 3) are fulfilled, but

$$\int_a^b p_i(t)\,dt = +\infty \ \ (i=0,1),$$

then the equation (1) has a nonintegrable singularity in the time variable at the points $t=a$ and $t=b$ (at the point $t=a$).

As an example, we consider the differential equation

$$u'' = \sum_{k=1}^{n} \frac{f_k(t)}{q(u)}\, u^{\lambda_k} + \frac{f_0(t)}{q(u)} \tag{9}$$

with the nonlocal boundary conditions of one of the following three types:

$$u(a) = \int_a^b h_1(u(s))\, d\varphi(s), \quad u(b) = \int_a^b h_2(u(s))\, d\psi(s); \tag{10}$$

$$u(a) = \int_a^b h_1(u(s))\, d\varphi(s), \quad u(b) = \int_a^{t_0} h_2(u(s))\, d\sigma(s); \tag{11}$$

$$u(a) = \int_a^b h_1(u(s))\, d\varphi(s), \quad u'(b) = \int_a^b h_2(u(s))\, d\psi(s), \tag{12}$$

where

$$0 < \lambda_i \le 1 \ \ (i = 1, \dots, n), \ \ t_0 \in]a, b[\,,$$

$f_k :]a, b[\to \mathbb{R}_+$ $(k = 0, 1, \dots, n)$ are measurable functions, $q :]0, +\infty[\to]0, +\infty[$ is a continuous, nondecreasing function, $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ are continuous functions, $\varphi : [a, b] \to [0, 1]$, $\psi : [a, b] \to [0, 1]$ and $\sigma : [a, t_0] \to [0, 1]$ are nondecreasing functions.

Theorems 1–3 result in the following corollaries.

Corollary 1. *If*

$$0 < \int_a^b (t-a)(b-t) f_0(t)\, dt < +\infty, \quad \int_a^b (t-a)(b-t) f_i(t)\, dt < +\infty \ \ (i = 1, \dots, n), \tag{13}$$

$$\lim_{x \to +\infty} \frac{h_i(x)}{x} < 1 \ \ (i = 1, 2),$$

and

$$q(+\infty) = +\infty, \tag{14}$$

then the problem (9), (10) *has at least one positive solution.*

Corollary 2. *If along with* (13) *and* (14) *the conditions*

$$\lim_{x \to +\infty} \frac{h_1(x)}{x} < 1, \ \ h_2(x) \le x \ \ \text{for} \ \ x > 0$$

are fulfilled, then the problem (9), (11) *has at least one positive solution.*

Corollary 3. *If*

$$0 < \int_a^b (t-a) f_0(t)\, dt < +\infty, \quad \int_a^b (t-a) f_i(t)\, dt < +\infty \ \ (i = 1, 2),$$

$$\limsup_{x \to +\infty} \frac{h_1(x)}{x} + (b-a) \limsup_{x \to +\infty} \frac{h_2(x)}{x} < 1,$$

and the condition (14) *is fulfilled, then the problem* (9), (12) *has at least one positive solution.*

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