# Nonlocal in Time Problems for Semilinear Multidimensional Wave Equations

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In the space $\mathbb{R}^{n+1}$ of variables $x=(x_1,\dots,x_n)$ and $t$, in the cylindrical domain $D_T=\Omega\times(0,T)$, where $\Omega$ is a Lipschitz domain in $\mathbb{R}^n$, consider a nonlocal problem on finding a solution $u(x,t)$ of the following equation

$$L_\lambda u:=\frac{\partial^2 u}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}+\lambda f(x,t,u)=F(x,t),\ \ (x,t)\in D_T, \tag{1}$$

satisfying the homogeneous boundary condition for the side boundary $\Gamma:=\partial\Omega\times(0,T)$ of the cylinder $D_T$

$$u\big|_\Gamma=0, \tag{2}$$

the initial condition

$$u(x,0)=\varphi(x),\ \ x\in\Omega, \tag{3}$$

and the nonlocal condition

$$K_\mu u_t:=u_t(x,0)-\sum_{i=1}^{l}\mu_i u_t(x,t_i)=\psi(x),\ \ x\in\Omega, \tag{4}$$

where $f$, $F$, $\varphi$ and $\psi$ are given functions; $t_i=const$, $0<t_1<t_2<\dots<t_l\le T$; $\lambda$ and $\mu$ are given constants and $n\ge 2$, $l\ge 1$.

**Definition.** Let $f\in C(\overline{D}_T\times\mathbb{R})$, $F\in L_2(D_T)$, $\varphi\in \overset{\circ}{W}{}_2^1(\Omega)$, $\psi\in L_2(\Omega)$. We call function $u$ a strong generalized solution of the problem (1)–(4) of the class $W_2^1$, if $u\in \overset{\circ}{W}{}_2^1(D_T,\Gamma):=\{w\in W_2^1(D_T):w|_\Gamma=0\}$ and there exists a sequence of functions $u_m\in \overset{\circ}{C}{}^2(\overline{D}_T,\Gamma):=\{w\in C^2(\overline{D}_T):w|_\Gamma=0\}$ such that $u_m\to u$ in the space $\overset{\circ}{W}{}_2^1(D_T,\Gamma)$, $L_\lambda u_m\to F$ in the space $L_2(D_T)$, $u_m|_{t=0}\to\varphi$ in the space $\overset{\circ}{W}{}_2^1(\Omega)$, and $K_\mu u_{mt}\to\psi$ in the space $L_2(\Omega)$.

It is obvious that a classical solution of the problem (1)–(4) of the space $C^2(\overline{D}_T)$ represents a strong generalized solution of this problem of the class $W_2^1$.

Let

$$g(x,t,u)=\int_0^u f(x,t,s)\,ds,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}. \tag{5}$$

Consider the following conditions imposed on functions $f=f(x,t,u)$ and $g=g(x,t,u)$ from (1) and (5)

$$f\in C(\overline{D}_T\times\mathbb{R}),\ \ |f(x,t,u)|\le M_1+M_2|u|^\alpha,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}, \tag{6}$$

$$g(x,t,u)\ge -M_3,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}, \tag{7}$$

$$g_t\in C(\overline{D}_T\times\mathbb{R}),\ \ g_t(x,t,u)\le M_4,\ \ (x,t,u)\in\overline{D}_T\times\mathbb{R}, \tag{8}$$

where $M_i=const\ge 0$, $1\le i\le 4$; $\alpha=const\ge 0$.

**Remark 1.** Let us consider some classes of functions $f=f(x,t,u)$ frequently encountered in applications and which satisfy the conditions (6), (7) and (8):

1. $f(x,t,u) = f_0(x,t)\beta(u)$, where $f_0, \frac{\partial}{\partial t} f_0 \in C(\overline{D}_T)$ and $\beta \in C(\mathbb{R})$, $|\beta(u)| \leq \widetilde{M}_1 + \widetilde{M}_2|u|^\alpha$, $\widetilde{M}_i = const \geq 0$, $\alpha = const \geq 0$. In this case $g(x,t,u) = f_0(x,t) \int\limits_0^u \beta(s)\, ds$ and when $f_0 \geq 0$, $\frac{\partial}{\partial t} f_0 \leq 0$, $\int\limits_0^u \beta(s)\, ds \geq -M$, $M = const \geq 0$, the conditions (6), (7) and (8) will be fulfilled.

2. $f(x,t,u) = f_0(x,t)|u|^\alpha \operatorname{sign} u$, where $f_0, \frac{\partial}{\partial t} f_0 \in C(\overline{D}_T)$ and $\alpha > 1$. In this case $g(x,t,u) = f_0(x,t)\frac{|u|^{\alpha+1}}{\alpha+1}$ and when $f_0 \geq 0$, $\frac{\partial}{\partial t} f_0 \leq 0$, the conditions (6), (7) and (8) will be also fulfilled.

**Theorem 1.** *Let $\lambda > 0$, $\sum\limits_{i=1}^{l} |\mu_i| < 1$, $F \in L_2(D_T)$, $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$; the conditions* (6), (7), *and* (8) *be fulfilled. Then, if the exponent of nonlinearity $\alpha$ in the condition* (6) *satisfies the inequality $\alpha < \frac{n+1}{n-1}$, then the problem* (1)–(4) *has at least one strong generalized solution of the class $W_2^1$.*

On the function $f$ in the equation (1) let us impose the following additional requirements

$$f, f'_u \in C(\overline{D}_T \times \mathbb{R}), \ \ |f'_u(x,t,u)| \leq a + b|u|^\gamma, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{9}$$

where $a, b, \gamma = const \geq 0$.

It is obvious that from (9) we have the condition (6) for $\alpha = \gamma + 1$, and when $\gamma < \frac{2}{n-1}$, we have $\alpha = \gamma + 1 < \frac{n+1}{n-1}$.

**Theorem 2.** *Let $\lambda > 0$, $\sum\limits_{i=1}^{l} |\mu_i| < 1$, $F \in L_2(D_T)$, $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$ and the condition* (9) *be fulfilled for $\gamma < \frac{2}{n-1}$, and also hold the conditions* (7), (8). *Then there exists a positive number $\lambda_0 = \lambda_0(F, f, \varphi, \psi, \mu, D_T)$ such that for $0 < \lambda < \lambda_0$ the problem* (1)–(4) *can not have more than one strong generalized solution of the class $W_2^1$.*

**Remark 2.** Note that if condition $\sum\limits_{i=1}^{l} |\mu_i| < 1$ is violated, as shown by specific examples, even in the linear case, i.e. for $f = 0$, the homogeneous problem corresponding to (1)–(4) may have finite or even an infinite set of independent solutions.

**Remark 3.** In the case, when the condition (7) is violated, the problem (1)–(4) may have no strong generalized solution of the class $W_2^1$. Indeed, let us consider the following condition imposed on function $f$

$$f(x,t,u) \leq -|u|^p, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}; \ \ p = const > 1. \tag{10}$$

**Theorem 3.** *Let $f \in C(\overline{D}_T \times \mathbb{R})$ satisfy the condition* (6), *when $0 \leq \alpha < \frac{n+1}{n-1}$ and the condition* (10)*; $\lambda > 0$, function $F^0 \in L_2(D_T)$, $F^0 \geq 0$, $\|F^0\|_{L_2(D_T)} \neq 0$ and $F = \gamma F^0$, $\gamma = const > 0$. Then there exists the number $\gamma_0 = \gamma_0(F^0, \alpha, p, \lambda) > 0$ such that for $\gamma > \gamma_0$ the problem* (1)–(4) *does not have a strong generalized solution of the class $W_2^1$.*