Precise Baire Characterization of the Lyapunov Exponents of Families of Morphisms of Metrized Vector Bundles with a Given Base

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In [1] V. M. Millionschikov introduced the concept of the Lyapunov exponents of families of morphisms of metrized vector bundles.

Let (E, p, B) be a vector bundle with fixed Riemann metric, fiber $\mathbb{R}^n$ and let the base B be a complete metric space. Let $|\cdot|$ denote norm in the fibers induced by the Riemann metric of a bundle. Suppose that for each $m \in \mathbb{N}$ a morphism $(X(m), \chi(m)) : (E, p, B) \to (E, p, B)$ is defined such that mappings $X(m)$ are non-degenerate on fibers. In other words, for each $m \in \mathbb{N}$ continuous mappings $X(m) : E \to E$ and $\chi(m) : B \to B$ are given such that $X(m) \circ p = p \circ \chi(m)$ and the restriction $X(m)|_{p^{-1}(b)}$ of mapping $X(m)$ to the fiber $p^{-1}(b)$ is a non-singular linear mapping (denote this restriction by $X(m, b)$). Suppose further that there is a function $a(\cdot) : B \to [0, +\infty)$ such that for each $m \in \mathbb{N}$ the inequality $\max\{\|X(m, b)\|, \|X^{-1}(m, b)\|\} \leqslant \exp(m \cdot a(b))$ holds, where $\|\cdot\|$ is an operator norm. Such a vector bundle is called n-dimensional Millionschikov bundle.

By $\mathfrak{M}_n$ denote the collection of all n-dimensional Millionschikov bundles, and by $\mathfrak{M}_n(B)$ denote the collection of all n-dimensional Millionschikov bundles with base B.

Then k-th Lyapunov exponent λ_k, $k = 1, \ldots, n$, of family of morphisms of Millionschikov bundle is defined [1] by:

$$\lambda_k(b) \stackrel{\text{def}}{=} \min_{V \in G_{n-k+1}(\mathbb{R}^n)} \ \max_{\xi \in V, |\xi|=1} \ \overline{\lim_{m \to +\infty}} \, m^{-1} \ln |X(m, b)\xi|,$$

where $\mathbb{R}^n$ is a fiber $p^{-1}(b)$ and $G_{n-k+1}(\mathbb{R}^n)$ is a Grassmannian manifold of $(n-k+1)$-dimensional lineals in $\mathbb{R}^n$. It follows from the definition that exponent $\lambda_k(\cdot)$ is a function $B \to \mathbb{R}$ and $\lambda_1(b) \geqslant \cdots \geqslant \lambda_n(b)$ for all $b \in B$.

Consider the question: what is the precise characterization of Lyapunov exponents of families of morphisms of Millionschikov bundles as functions on the base of the bundle? V. M. Millionschikov [1] proved that every function $\lambda_k(\cdot) : B \to \mathbb{R}$ is a function of the second Baire class.

M. I. Rakhimberdiev [2] proved that the number of Baire class in the statement above cannot be reduced. A. N. Vetokhin [3, 4] proved that the Lyapunov exponents are functions of class $({}^*, G_\delta)$ in spaces $\mathcal{M}_n^{\mathrm{c}}$ and $\mathcal{M}_n^{\mathrm{u}}$. Here $\mathcal{M}_n^{\mathrm{c}}$ and $\mathcal{M}_n^{\mathrm{u}}$ are the spaces of linear differential systems with piecewise continuous coefficients bounded on the half-line and with the topology of compact and uniform convergence, respectively.

A complete answer to the question formulated for the class $\mathfrak{M}_n$ of all Millionschikov bundles is given in the article [5]. In this article it is shown that all exponents $\lambda_k(\cdot) : B \to \mathbb{R}$ belong to a class $({}^*, G_\delta)$, have upper semi-continuous minorant and satisfy inequalities $\lambda_1(b) \geqslant \cdots \geqslant \lambda_n(b)$ for all $b \in B$; and also there is shown sufficiency of this three conditions. The theorem below shows that a similar result holds for each class $\mathfrak{M}_n(B)$ for any $m \in \mathbb{N}$ and a complete metric space B.

Recall that a real function belongs to a class $({}^*, G_\delta)$ if the inverse image of any interval $[r, +\infty)$, $r \in \mathbb{R}$ is a G_δ-set. Let $m(\cdot)$ and $\lambda(\cdot)$ be functions $B \to \mathbb{R}$, then function $m(\cdot)$ is called minorant of a function $\lambda(\cdot)$ if $\lambda(b) \geqslant m(b)$ for all $b \in B$.

Theorem. *Every Lyapunov exponent λ_k, $k = 1, \ldots, n$, of a family of morphisms of Millionschikov bundle is a function of class $({}^*, G_\delta)$ having upper semi-continuous minorant and satisfying $\lambda_1(b) \geqslant \cdots \geqslant \lambda_n(b)$ for all $b \in B$.*

Conversely for every $n \in \mathbb{N}$ and a complete metric space B there exists an n-dimensional vector bundle $\mathfrak{R}$ with the base B and a fiber $\mathbb{R}^n$ such that for any set $(f_1(\cdot), \ldots, f_n(\cdot))$ of functions $B \to \mathbb{R}$

of a class $({}^*, G_\delta)$ *having upper semi-continuous minorant and satisfying* $f_1(b) \geqslant \cdots \geqslant f_n(b)$ *for all* $b \in B$*, there exists such a family of morphisms of* $\mathfrak{R}$ *that* $\mathfrak{R}$ *together with this family of morphisms is a Millionschikov bundle and set of Lyapunov exponents* $\lambda_1, \ldots, \lambda_k$ *of this bundle coincide with the set* $(f_1, \ldots, f_n)$ *i.e.* $\lambda_k(b) = f_k(b)$ *for all* $b \in B$ *and* $k \in \{1, \ldots, n\}$*.*

References

[1] V. M. Millionschikov, Birov classes of functions and Lyapunov indices. I. (Russian) *Differentsial'nye Uravneniya* **16** (1980), No. 8, 1408–1416; translation in *Differential Equat.* **16** (1981), No. 8, 902–907.

[2] M. I. Rakhimberdiev, A Baire class of Lyapunov exponents. (Russian) *Mat. Zametki* **31** (1982), No. 6, 925–931.

[3] A. N. Vetokhin, Lebesque Sets of Lyapunov Exponents. (Russian) *Differentsial'nye Uravneniya* **37** (2001), No. 6, 849; translation in *Differential Equat.* **37** (2001), No. 6, 892.

[4] A. N. Vetokhin, On Lebesque Sets of Lyapunov Exponents. (Russian) *Differentsial'nye Uravneniya* **38** (2002), No. 11, 1567; translation in *Differential Equat.* **38** (2002), No. 11, 1665.

[5] M. V. Karpuk, Precise Baire characterization of the Lyapunov exponents of families of morphisms of metrized vector bundles. (Russian) *Dokl. National Acad. Sci. Belarus* **57** (2013), No 2. 11–16.