Exponential Stability of Linear Itô Equations with Delay and Azbelev's W-Transform

Ramazan I. Kadiev

Dagestan Research Center of the Russian Academy of Sciences & Department of Mathematics, Dagestan State University, Makhachkala, Russia
E-mail: `kadiev_r@mail.ru`

Arcady Ponosov

Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences, Norway
E-mail: `arkadi@umb.no`

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By $\mathbb{E}$ we denote the expectation on this probability space. The scalar stochastic process $\mathcal{B}$ is a Brownian motion on $(\mathcal{F}_t)_{t\geq 0}$ (see e.g. [6]).

Below we always assume that the real number p satisfies the conditions $1 \leq p < \infty$.

Consider the following scalar stochastic delay differential equation

$$dx(t) = \big(-a(t)x(t) - b(t)x(h(t))\big)\,dt + c(t)x(H(t))\,d\mathcal{B}(t) \quad (t \geq 0), \tag{1}$$

equipped with the prehistory condition

$$x(\nu) = \varphi(\nu) \quad (\nu < 0), \tag{2}$$

where φ is a $\mathcal{F}_0$-measurable stochastic process which almost surely (a. s.) has trajectories from L^∞. The functions a, b, c, h, H in (1) are all Lebesgue-measurable, a, b are, in addition, locally integrable, c is locally square-integrable, $h(t) \leq t$, $H(t) \leq t$ for $t \in [0, \infty)$ a.s., $\operatorname*{vraisup}_{t\geq 0}(t - h(t)) < \infty$, $\operatorname*{vraisup}_{t\geq 0}(t - H(t)) < \infty$.

The initial condition for the equation (1) is given by

$$x(0) = x_0, \tag{3}$$

where x_0 is a $\mathcal{F}_0$-measurable scalar random variable.

Definition. The zero solution of the equation (1) is called exponentially Lyapunov $2p$-stable (with respect to the prehistory data (2) and the initial data (3)) if there exist positive numbers $\bar{c}$ and β such that

$$\mathbb{E}|x(t, x_0, \varphi)|^{2p} \leq \bar{c}\Big(\mathbb{E}|x_0|^{2p} + \operatorname*{vraisup}_{\nu<0} \mathbb{E}|\varphi(\nu)|^{2p}\Big) \exp\{-\beta t\} \quad (t \geq 0) \tag{4}$$

for any $\mathcal{F}_0$-measurable scalar random variable x_0 and any $\mathcal{F}_0$-measurable stochastic process φ, which a.s. has trajectories belonging to L^∞.

In the theorems below we use the universal constants c_p $(1 \leq p < \infty)$ from the celebrated Burkholder–Davis–Gandy inequalities to estimate stochastic integrals. From [7] it is e.g. known that $c_p = 2\sqrt{12}p$. However, other sources give other values (see e.g. [6]). All these values are not optimal for our purposes. For instance, in our theorems we may assume that $c_1 = 1$, as we estimate $\sup_t \mathbb{E}|x(t)|^2$, and not $\mathbb{E}\sup_t |x(t)|^2$, as in the Burkholder–Davis–Gandy inequalities.

Given a function $h(t)$ $(t \in [T, \infty))$, we construct the function $h^T(t)$ $(t \in [T, \infty))$ in the following way:

$$h^T(t) = \begin{cases} h(t) & \text{if } h(t) \geq T, \\ T & \text{if } h(t) < T. \end{cases}$$

Theorem 1. *Assume that there exist numbers $a_0 > 0$, $\gamma_i > 0$, $i = 1, 2$, $T \in [0, \infty)$ such that one of the following conditions holds:*

(1) $a(t) \geq a_0$, $|b(t)| \leq \gamma_1 a(t)$, $(c(t))^2 \leq 2\gamma_2 a(t)$ $(t \geq T)$ *a.s.*,

(2) $a(t) + b(t) \geq a_0$, $|b(t)| \Big[\int\limits_{h^T(t)}^{t} \big(|a(s)| + |b(s)|\big)\, ds + c_p \Big(\int\limits_{h^T(t)}^{t} (c(s))^2\, ds \Big)^{0.5} \Big] \leq$

$$\leq \gamma_1(a(t) + b(t)), \quad (c(t))^2 \leq 2\gamma_2(a(t) + b(t)) \ \ (t \geq T) \ \ a.s.$$

If, in addition, $\gamma_1 + c_p\sqrt{\gamma_2} < 1$, then the solutions of the equation (1) *satisfy the estimate* (4) *for some $\beta > 0$.*

Theorem 2. *Assume that there exists a number $T \in [0, \infty)$ such that the coefficients in* (1) *satisfy*

$$a(t) = Ar(t), \quad b(t) = Br(t), \quad c(t) = C\sqrt{r(t)}\,, \quad r(t) \geq r_0 > 0 \ \ (t \in [T, \infty)) \ \ a.s.$$

Assume also that one of the following conditions holds:

(1) $A > 0$, $|B|/A + c_p|C|/\sqrt{2A} < 1$,

(2) $A + B > 0$, $\lim\limits_{t\to\infty} \sup\limits_{T \leq \tau \leq t} |B| \Big[\int\limits_{h^T(\tau)}^{\tau} r(s)\, ds(|A| + |B|) + c_p \Big(\int\limits_{h^T(\tau)}^{\tau} r(s)\, ds \Big)^{0.5} |C| \Big] \Big/ (A + B) +$

$$+ c_p|C|/\sqrt{2(A + B)} < 1,$$

(3) $A > 0$, $B > 0$, $\lim\limits_{t\to\infty} \sup\limits_{T \leq \tau \leq t} B \Big[\int\limits_{h^T(\tau)}^{\tau} r(s)\, ds + c_p \Big(\int\limits_{h^T(\tau)}^{\tau} r(s) ds \Big)^{0.5} |C| \Big/ (A + B) \Big] +$

$$+ c_p|C|/\sqrt{2(A + B)} < 1.$$

Then the solutions of the equation (1) *satisfy the estimate* (4) *for some $\beta > 0$.*

The proof of the results is based on Azbelev's W-transform of the equation (1) (see e.g. [1, 2, 5]), which uses the so-called 'reference equation' and which in our case is defined as follows:

$$dx(t) = \big[a(t)x(t) + g_1(t)\big]\, dt + g_2(t)\, d\mathcal{B}(t) \ \ (t \geq 0),$$

where g_1 is an $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process with a.s. locally integrable trajectories, g_2 is an $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process with a.s. locally square-integrable trajectories, and the other parameters are defined in (1). This is an ordinary differential equation with stochastic perturbations. The W-method works if the reference equation is exponentially stable and if a certain integral operator, which combines the equation (1) with the reference equation, is invertible. To describe this idea in more detail, we introduce the notation $Z(t) = (t, \mathcal{B}(t))'$ and unify the equation (1) with the prehistory condition (2) into a single stochastic functional differential equation

$$dx(t) = \left[(Vx)(t) + f(t)\right] dZ(t) \ \ (t \geq 0), \tag{5}$$

where

$$(Vx)(t) = \big((V_1x)(t), (V_2x)(t)\big), \quad (V_1x)(t) = a(t)x(t) + b(t)(S_hx)(t), \quad (V_1x)(t) = c(t)(S_Hx)(t),$$
$$f(t) = \big(f_1(t), f_2(t)\big), \quad f_1(t) = b(t)\varphi_h(t), \quad f_2(t) = c(t)\varphi_H(t),$$
$$(S_\gamma x)(t) = \begin{cases} x(\gamma(t)) & \text{if } \gamma(t) \geq 0, \\ 0 & \text{if } \gamma(t) < 0, \end{cases} \qquad \varphi_\gamma(t) = \begin{cases} 0 & \text{if } \gamma(t) \geq 0, \\ \varphi(\gamma(t)) & \text{if } \gamma(t) < 0. \end{cases}$$

Similarly, we can rewrite the reference equation as follows:

$$dx(t) = \left[(Qx)(t) + g(t)\right] dZ(t) \ \ (t \geq 0),$$

where $(Qx)(t) = (ax(t), 0)$, $g(t) = (g_1(t), g_2(t))$.

It is well-known (see e.g. [5]) that the latter equation admits the integral representation $x(t) = U(t)x(0) + (Wg)(t)$ $(t \geq 0)$, where $U(t)$ is the fundamental matrix of the associated homogeneous equation, W is the integral operator

$$(Wg)(t) = \int_0^t C(t,s)g(s)\, dZ(s) \ \ (t \geq 0),$$

and the function $C(t,s)$ is defined on $G := \{(t,s) :\ t \in [0,\infty),\ 0 \leq s \leq t\}$.

Assume that U and W satisfy the following conditions:

R1. $|U(t)| \leq \overline{c}$, where $\overline{c} \in \mathbf{R}_+$.

R2. $|C(t,s)| \leq \overline{\overline{c}} \exp\{-\alpha(t-s)\}$ for some $\alpha > 0$, $\overline{\overline{c}} > 0$.

The integral representation gives then rise to the W-transform, which is applied to the equation (5) in the following manner:

$$dx(t) = \left[(Qx)(t) + ((V-Q)x)(t) + f(t)\right] dZ(t) \ \ (t \geq 0),$$

or, alternatively,

$$x(t) = U(t)x(0) + (W(V-Q)x)(t) + (Wf)(t) \ \ (t \geq 0).$$

Denoting $W(V-Q) = \Theta$, we obtain the operator equation

$$((I - \Theta)x)(t) = U(t)x(0) + (Wf)(t) \ \ (t \geq 0).$$

Finally, we put

$$M_{2p,T} \equiv \left\{x :\ x \in C,\ \|x\|_{M_{2p,T}} := \Big(\sup_{t \geq T} \mathbb{E}|x(t)|^{2p}\Big)^{1/2p} < \infty\right\},$$

where C denotes the set of all $(\mathcal{F}_t)_{t \geq T}$-adapted stochastic processes with a. s. continuous trajectories.

The main idea of the W-transform approach is to prove invertibility of the operator $I - \Theta$ in the space $M_{2p,T}$, which would imply the exponential $2p$-stability of the solutions of the equation (1). This is summarized in the following lemma:

Lemma. *Let the reference equation satisfy the conditions* **R1**–**R2**. *Assume that the operator* $(I-\Theta): M_{2p,T} \to M_{2p,T}$ *has a bounded inverse for some* $T \geq 0$. *Then the solutions of the equation* (1) *satisfy the estimate* (4) *for some* $\beta > 0$.

The proof of the Theorems 1 and 2 consists then in checking the assumptions of the lemma.

Remark. More details about the theory and applications of the W-transform to deterministic functional differential equations can be found in [1, 2]. The stochastic version of this transform, an outline of which is presented above, was comprehensively studied in [3]. An alternative yet similar version of the stochastic W-transform, where the integral substitution is performed in a different manner, was suggested in [4] on the basis of the deterministic results obtained in [1, 2].

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