The Mixed Problem for the Semilinear Wave Equation with a Nonlinear Boundary Condition

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In the plane of independent variables x and t in the domain $D_T := \{(x,t) \in \mathbb{R}^2 : \ 0 < x < l, \ 0 < t < T\}$ we consider a mixed problem of finding a solution $u(x,t)$ for the semilinear wave equation of the type

$$u_{tt} - u_{xx} + g(u) = f(x,t), \ \ (x,t) \in D_T, \tag{1}$$

satisfying the following initial

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \ \ 0 \le x \le l, \tag{2}$$

and boundary conditions

$$u_x(0,t) = F[u(0,t)] + \beta(t), \quad u(l,t) = \nu(t), \ \ 0 \le t \le T, \tag{3}$$

where f, φ, ψ, β, ν, g and F are the given and u is an unknown real functions.

Let the conditions of smoothness

$$\begin{gathered} f \in C^1(\overline{D}_T), \ \ g, \ F \in C^1(\mathbb{R}), \\ \varphi \in C^2([0,l]), \ \ \psi \in C^1([0,l]), \ \ \beta \in C^1([0,l]), \ \ \nu \in C^2([0,l]) \end{gathered} \tag{4}$$

and the agreement

$$\begin{gathered} \varphi(0) = 0, \quad \varphi(l) = 0, \quad \psi(0) = 0, \quad \varphi'(0) = F(0) + \beta(0), \quad \psi'(0) = \beta'(0), \\ \varphi(l) = \nu(0), \quad \psi(l) = \nu'(0), \quad \nu''(0) - \varphi''(l) + g(0) = f(l,0) \end{gathered} \tag{5}$$

be fulfilled.

Note that the nonlinear boundary condition of the type (3) arises, for example, in describing the process of longitudinal string oscillations if one of its ends is elastically fixed, when tension on that end is a nonlinear function of displacement, and also in processes taking place in distributed self-oscillating systems.

Consider the conditions

$$G(g;s) := \int\limits_0^s g(s_1)\,ds_1 \ge -M_1 s^2 - M_2, \quad \int\limits_0^s F(s_1)\,ds_1 \ge -M_3 \ \ \forall\, s \in \mathbb{R}, \tag{6}$$

where $M_i := const \ge 0$, $1 \le i \le 3$.

The following theorem is valid.

Theorem 1. *Let the conditions* (4)–(6) *be fulfilled. Then there exists a unique classical solution of the problem* (1)–(3).

Remark. In the case if at least one of the conditions (6) imposed on the nonlinear functions g and F is violated, then relying on the comparison theorem, we can distinguish those data classes of the problem (1)–(3) for which the problem is globally solvable in one case or has blow up solution in the other case.

Before formulating the comparison theorem, we consider nonlinear mixed problems in the following statement:

$$\begin{gathered} u_{tt}-u_{xx}+g_i(u)=f_i(x,t),\ \ (x,t)\in D_T,\\ u(x,0)=\varphi_i(x),\quad u_t(x,0)=\psi_i(x),\ \ 0\le x\le l,\\ u_x(0,t)=F_i[u(0,t)]+\beta_i(t),\quad u(l,t)=\nu_i(t),\ \ 0\le t\le T, \end{gathered}\tag{7}$$

where the data of the problem satisfy the corresponding smoothness and agreement conditions.

Theorem 2. *Let u_1 and u_2 be classical solutions of the problem* (7) *for $i=1$ and $i=2$, respectively. Then, if*

$$\begin{gathered} g_1'(s)\le 0\ \ or\ \ g_2'(s)\le 0;\quad g_1(s)\ge g_2(s),\ \ s\in\mathbb{R},\\ f_1(x,t)\le f_2(x,t),\ \ (x,t)\in \overline{D}_T,\\ \varphi_1(x)\le\varphi_2(x),\quad \varphi_1'(x)\ge\varphi_2'(x),\quad \psi_1(x)\le\psi_2(x),\ \ 0\le x\le l,\\ F_1'(s)\le 0\ \ or\ \ F_2'(s)\le 0;\quad F_1(s)\ge F_2(s),\ \ s\in\mathbb{R},\\ \beta_1(t)\ge\beta_2(t),\quad \nu_1(t)\le\nu_2(t),\ \ 0\le t\le T, \end{gathered}$$

then

$$u_1(x,t)\le u_2(x,t),\ \ (x,t)\in\overline{D}_T.$$

Let the functions g, f, φ, ψ, F, β and ν satisfy the conditions

$$\begin{gathered} g\ge 0,\ \ g(s)=0,\ s\ge 0;\ \ f=0;\ \ \varphi\ge 0,\ \varphi'\le 0,\ \psi\ge 0,\ \psi\not\equiv 0;\ \ \beta=0,\ \nu\ge 0,\\ F'\le 0,\quad F(0)=0,\quad F(s)\ge -\delta s^\alpha,\ \ s\ge 0,\ \ \delta>0,\ \ \alpha>1,\\ \frac{1}{\delta(\alpha-1)[\varphi(0)+k_1 l]^{\alpha-1}}>T, \end{gathered}\tag{8}$$

where $k_1:=\|\psi\|_{C([0,l])}$.

Theorem 3. *Let the conditions* (8) *be fulfilled. Then the problem* (1)–(3) *has a unique classical solution.*

Let now the conditions

$$\begin{gathered} g\le 0;\quad f\ge 0,\ \ \beta\le 0,\\ F(s)\le -\delta|s|^\alpha s,\quad \delta:=const>0,\quad \alpha:=const>0,\ \ s\in\mathbb{R},\\ \varphi(0)>k_2 l,\quad T^*:=\frac{1}{\delta\alpha[\varphi(0)-k_2 l]^{\alpha-1}}\le T \end{gathered}\tag{9}$$

be fulfilled, where $k_2:=\max\limits_{0\le x\le l}|\varphi'(x)+\psi(x)|$.

Theorem 4. *Let the conditions* (9) *be fulfilled. Then there is $T_*\in(0,T^*]$ such that in the domain D_{T_*} there exist a unique solution of the problem* (1)–(3) *of the class $C^2(\overline{D}_{T_*}\setminus t=T_*)$ which blows up, i.e., satisfies the condition*

$$\lim_{T\to T_*-0}\Big(\|u\|_{C(\overline{D}_T)}+\|u_t\|_{C(\overline{D}_T)}\Big)=\infty.$$

Note that if the above conditions for the problem data are violated, the local solvability with respect to t remains valid. In this case it is sufficient to require of the problem data that $f\in C^1(\overline{D}_T)$, $g,F\in C^1(\mathbb{R})$, $\varphi\in C^2([0,l])$, $\psi\in C^1([0,l])$, $\beta\in C^1([0,T])$, $\nu\in C^2([0,T])$.