Asymptotic Analysis of Positive Decreasing Solutions of First Order Nonlinear Functional Differential Systems in the Framework of Regular Variation

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We consider the system of first order nonlinear functional differential equations of the type

$$\begin{cases} x'(t) + p_1(t)x(g_1(t))^{\alpha_1} + q_1(t)y(h_1(t))^{\beta_1} = 0, \\ y'(t) + p_2(t)x(g_2(t))^{\alpha_2} + q_2(t)y(h_2(t))^{\beta_2} = 0, \end{cases} \tag{A}$$

where α_i, β_i are positive constants, $p_i(t)$, $q_i(t)$ are positive continuous functions on $[a,\infty)$, $a > 0$, and $g_i(t)$, $h_i(t)$ are positive continuous functions on $[a,\infty)$ such that $\lim\limits_{t\to\infty} g_i(t) = \lim\limits_{t\to\infty} h_i(t) = \infty$, $i = 1,2$. By a positive solution of (A) we mean a vector function $(x(t), y(t))$ whose components are positive on an interval of the form $[T,\infty)$ and satisfy the system (A) there. A positive solution $(x(t), y(t))$ is said to be *strongly decreasing* if $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} y(t) = 0$.

Our aim is to derive nontrivial information about the existence and the asymptotic behavior of strongly decreasing solutions of (A) by making an analysis of the problem in the framework of regular variation (in the sense of Karamata). By definition a measurable function $f : [a,\infty) \to (0,\infty)$ is called *regularly varying of index* $\rho \in \mathbf{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for } \ \forall\, \lambda > 0.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. If in particular $\rho = 0$, then the symbol SV is often used for $\mathrm{RV}(0)$, and its members are referred to as *slowly varying functions.*

The simplest cases of (A) are the diagonal system

$$x'(t) + p_1(t)x(g_1(t))^{\alpha_1} = 0, \quad y'(t) + q_2(t)y(h_2(t))^{\beta_2} = 0, \tag{$\mathrm{A_d}$}$$

and the cyclic system

$$x'(t) + q_1(t)y(h_1(t))^{\beta_1} = 0, \quad y'(t) + p_2(t)x(g_2(t))^{\alpha_2} = 0. \tag{$\mathrm{A_c}$}$$

Some of the main results for these systems follow.

Theorem 1. *Suppose that $\alpha_1 < 1$ and $\beta_2 < 1$, $p_1 \in \mathrm{RV}(\lambda_1)$ and $q_2 \in \mathrm{RV}(\mu_2)$, and*

$$\lim_{t\to\infty} \frac{g_1(t)}{t} = \lim_{t\to\infty} \frac{h_2(t)}{t} = 1. \tag{1}$$

System $(\mathrm{A_d})$ possesses strongly decreasing solutions $(x,y) \in \mathrm{RV}(\rho) \times \mathrm{RV}(\sigma)$ with $\rho < 0$ and $\sigma < 0$ if and only if

$$\lambda_1 + 1 < 0 \ \text{ and } \ \mu_2 + 1 < 0, \tag{2}$$

in which case ρ *and* σ *are given by*

$$\rho = \frac{\lambda_1 + 1}{1 - \alpha_1}, \quad \sigma = \frac{\mu_2 + 1}{1 - \beta_2}, \tag{3}$$

and the asymptotic behavior of any such solution $(x(t), y(t))$ *is governed by the unique decay law*

$$x(t) \sim X_1(t) = \Big(\frac{tp_1(t)}{-\rho}\Big)^{\frac{1}{1-\alpha_1}}, \quad y(t) \sim Y_1(t) = \Big(\frac{tq_2(t)}{-\sigma}\Big)^{\frac{1}{1-\beta_2}}, \quad t \to \infty. \tag{4}$$

Theorem 2. *Suppose that* $\alpha_2\beta_1 < 1$, $q_1 \in \mathrm{RV}(\mu_1)$ *and* $p_2 \in \mathrm{RV}(\lambda_2)$, *and*

$$\lim_{t\to\infty} \frac{g_2(t)}{t} = \lim_{t\to\infty} \frac{h_1(t)}{t} = 1. \tag{5}$$

System $(\mathrm{A_c})$ *possesses strongly decreasing solutions* $(x, y) \in \mathrm{RV}(\rho) \times \mathrm{RV}(\sigma)$ $\rho < 0$ *and* $\sigma < 0$ *if and only if*

$$\beta_1(\lambda_2 + 1) + \mu_1 + 1 < 0 \ \ \textit{and} \ \ \lambda_2 + 1 + \alpha_2(\mu_1 + 1) < 0, \tag{6}$$

in which case ρ *and* σ *are given by*

$$\rho = \frac{\beta_1(\lambda_2 + 1) + \mu_1 + 1}{1 - \alpha_2\beta_1}, \quad \sigma = \frac{\lambda_2 + 1 + \alpha_2(\mu_1 + 1)}{1 - \alpha_2\beta_1}, \tag{7}$$

and the asymptotic behavior of any such solution $(x(t), y(t))$ *is governed by the unique decay law*

$$x(t) \sim X_2(t) = \Big[\frac{tq_1(t)}{-\rho}\Big(\frac{tp_2(t)}{-\sigma}\Big)^{\beta_1}\Big]^{\frac{1}{1-\alpha_2\beta_1}}, \quad y(t) \sim Y_2(t) = \Big[\frac{tp_2(t)}{-\sigma}\Big(\frac{tq_1(t)}{-\rho}\Big)^{\alpha_2}\Big]^{\frac{1}{1-\alpha_2\beta_1}}, \tag{8}$$

as $t \to \infty$.

Here the symbol $\sim$ is used to mean the asymptotic equivalence of two positive functions:

$$f(t) \sim g(t), \ \ t \to \infty \iff \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

Turning to system (A), it is expected that if (A) can be regarded as a "small" perturbation of the diagonal system $(\mathrm{A_d})$ (resp. $(\mathrm{A_c})$), then it may possess strongly decreasing regularly varying solutions $(x(t), y(t))$ which behave like (4) (resp. (8)) as $t \to \infty$. The correctness of this expectation is shown by the following results.

Theorem 3. *Suppose that* $\alpha_1 < 1$ *and* $\beta_2 < 1$, *that* $p_1 \in \mathrm{RV}(\lambda_1)$ *and* $q_2 \in \mathrm{RV}(\mu_2)$, *and that* $g_1(t)$ *and* $h_2(t)$ *satisfy* (1). *Suppose moreover that* λ_1 *and* μ_2 *satisfy* (2) *and define* ρ *and* σ *by* (3). *If*

$$\lim_{t\to\infty} \frac{q_1(t)Y_1(h_1(t))^{\beta_1}}{p_1(t)X_1(g_1(t))^{\alpha_1}} = 0, \quad \lim_{t\to\infty} \frac{p_2(t)X_1(g_2(t))^{\alpha_2}}{q_2(t)Y_1(h_2(t))^{\beta_2}} = 0, \tag{9}$$

then system (A) *possesses a strongly decreasing regularly varying solution* $(x(t), y(t))$ *belonging to* $\mathrm{RV}(\rho) \times \mathrm{RV}(\sigma)$ *and enjoying the asymptotic behavior* (4).

Theorem 4. *Suppose that* $\alpha_2\beta_1 < 1$, *that* $q_1 \in \mathrm{RV}(\mu_1)$ *and* $p_2 \in \mathrm{RV}(\lambda_2)$, *and that* $g_2(t)$ *and* $h_1(t)$ *satisfy* (5). *Suppose moreover that the inequalities* (6) *hold, and define* ρ *and* σ *by* (7). *If*

$$\lim_{t\to\infty} \frac{p_1(t)X_2(g_1(t))^{\alpha_1}}{q_1(t)Y_2(h_1(t))^{\beta_1}} = 0, \quad \lim_{t\to\infty} \frac{q_2(t)Y_2(h_2(t))^{\beta_2}}{p_2(t)X_2(g_2(t))^{\alpha_2}} = 0, \tag{10}$$

then system (A) *possesses a strongly decreasing regularly varying solution* $(x(t), y(t))$ *belonging to* $\mathrm{RV}(\rho) \times \mathrm{RV}(\sigma)$ *and enjoying the asymptotic behavior* (8).

Remark 1. There are three possible types of strongly decreasing regularly varying solutions $(x(t), y(t))$ of (A):

(I) $(x, y) \in \mathrm{RV}(\rho) \times \mathrm{RV}(\sigma)$ with $\rho < 0$ and $\sigma < 0$;

(II) $(x, y) \in \mathrm{SV} \times \mathrm{RV}(\sigma)$ with $\sigma < 0$, or $(x, y) \in \mathrm{RV}(\rho) \times \mathrm{SV}$ with $\rho < 0$;

(III) $(x, y) \in \mathrm{SV} \times \mathrm{SV}$.

All of the above theorems are concerned with solutions of type (I). It should be noticed that for system ($\mathrm{A_d}$) with $\alpha_1 < 1$ and $\beta_2 < 1$ (resp. system ($\mathrm{A_c}$) with $\alpha_2\beta_1 < 1$) one can characterize the existence of solutions of types (II) and (III) (resp. solutions of type (II)) and determine their asymptotic behavior at infinity precisely.

Remark 2. We note that in Theorem 3 neither $p_2(t)$ nor $q_1(t)$ is assumed to be regularly varying, and neither $g_2(t)$ nor $h_1(t)$ is required to be asymptotic to t as $t \to \infty$, and the same is true of $p_1(t)$, $q_2(t)$, $g_1(t)$ and $h_2(t)$ in Theorem 4.

Remark 3. Assume in Theorems 3 and 4 that all $p_i(t)$, $q_i(t)$, $g_i(t)$ and $h_i(t)$ are regularly varying: $p_i \in \mathrm{RV}(\lambda_i)$, $q_i \in \mathrm{RV}(\mu_i)$, $g_i \in \mathrm{RV}(\gamma_i)$, $h_i \in \mathrm{RV}(\delta_i)$, $i = 1, 2$. Then, in the language of regular variation, condition (9) is interpreted as

$$\lim_{t\to\infty} t^{\mu_1+\beta_1\sigma\delta_1-(\lambda_1+\alpha_1\rho)} L(t) = 0, \quad \lim_{t\to\infty} t^{\lambda_2+\alpha_2\rho\gamma_2-(\mu_2+\beta_2\sigma)} M(t) = 0, \tag{11}$$

where $L(t)$ and $M(t)$ are slowly varying functions which can easily be computed from the slowly varying parts of $p_i(t)$ and $q_i(t)$. It is clear that the inequalities

$$\mu_1 + \beta_1\sigma\delta_1 < \lambda_1 + \alpha_1\rho, \quad \lambda_2 + \alpha_2\rho\gamma_2 < \mu_2 + \beta_2\sigma, \tag{12}$$

are sufficient for (11) to be satisfied regardless of $L(t)$ and $M(t)$, and hence (12) is often utilized as a convenient criterion for the existence of strongly decreasing solutions of system (A) which are regularly varying of negative indices. A similar interpretation of (10) in Theorem 4 can be made in terms of regular variation.