On Some Properties and Approximate Solution of One System of Nonlinear Partial Differential Equations

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Many applied problems leads to the following nonlinear system

$$\frac{\partial U}{\partial t}=\sum_{\alpha=1}^{p}\frac{\partial}{\partial x_\alpha}\Big(V_\alpha\,\frac{\partial U}{\partial x_\alpha}\Big), \tag{1}$$

$$\frac{\partial V_\alpha}{\partial t}=f_\alpha\Big(V_\alpha,\frac{\partial U}{\partial x_\alpha}\Big),\ \ \alpha=1,\dots,p, \tag{2}$$

where f_α are given functions. If $p=2$ and

$$f_\alpha\Big(V_\alpha,\frac{\partial U}{\partial x_\alpha}\Big)=-V_\alpha+g_\alpha\Big(V_\alpha\,\frac{\partial U}{\partial x_\alpha}\Big),\ \ 0<\gamma_0<g_\alpha(\xi_\alpha)\le G_0,\ \ \alpha=1,2, \tag{3}$$

where g_α are given sufficiently smooth functions and γ_0, G_0 are constants, the system (1)–(3) describes the vein formation in meristematic tissues of young leaves [1]. Investigations for one-dimensional analog of model (1)–(3) are carried out in [2]. The large theoretical and practical importance of the investigation and construction of approximate solutions of the boundary value problems for systems (1)–(3) is pointed out in the above-mentioned works [1, 2]. In the direction of biological modeling it is necessary to note work [3], where many mathematical models of similar diffusion processes are also presented and discussed.

There are some effective algorithms for solving the multi-dimensional problems (see, for example, [4] and references therein). These algorithms mainly belong to the methods of splitting-up or sum approximation according to their approximate properties. Some schemes of the variable directions are constructed and studied in [5] and in a number of other works by this author too. Some questions of construction and investigation of the discrete analogs for systems of the type (1)–(3) are discussed in [6–9] and in a number of other works as well.

Let us consider the following two-dimensional initial-boundary value problem:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x_1}\Big(V_1\,\frac{\partial U}{\partial x_1}\Big)+\frac{\partial}{\partial x_2}\Big(V_2\,\frac{\partial U}{\partial x_2}\Big), \tag{4}$$

$$\frac{\partial V_\alpha}{\partial t}=-V_\alpha+g_\alpha\Big(V_\alpha\,\frac{\partial U}{\partial x_\alpha}\Big),\ \ \alpha=1,2, \tag{5}$$

$$U(x,0)=U_0(x),\quad V_\alpha(x,0)=V_{\alpha0}(x),\ \ x\in\overline{\Omega},\ \ \alpha=1,2, \tag{6}$$

$$U(x,t)=0,\ \ (x,t)\in\partial\Omega\times[0,T]. \tag{7}$$

Here $x=(x_1,x_2)$, $\Omega=(0,1)\times(0,1)$, $\partial\Omega$ is the boundary of the domain Ω, T is some fixed positive number, U_0, $V_{\alpha0}$, g_α are given sufficiently smooth functions, such that

$$V_{\alpha0}(x)\ge\delta_0,\ \ x\in\overline{\Omega}, \tag{8}$$

$$\gamma_0\le g_\alpha(\xi_\alpha)\le G_0,\quad \big|g'_\alpha(\xi_\alpha)\big|\le G_1,\ \ \xi_\alpha\in R,\ \ \alpha=1,2, \tag{9}$$

where δ_0, γ_0, G_0, G_1 are some positive constants.

Suppose that all necessary consistence conditions are satisfied and there exists a sufficiently smooth solution of the problem (4) - (7). It should be noted that the uniqueness of a solution of the problem (4)–(7) is studied in [6].

Under the conditions (8), (9) from (5), (6) in $\overline{Q} = \overline{\Omega} \times [0, T]$ we have

$$\delta_0 \le V_\alpha(x,t) \le \Delta_0, \quad \alpha = 1,2, \tag{10}$$

where Δ_0 is a positive constant. Using (5), (9), (10) we get the estimates

$$\left|\frac{\partial V_\alpha}{\partial t}\right| \le C, \quad (x,t) \in \overline{Q}, \quad \alpha = 1,2.$$

Here and below C is a positive constant.

Introduce on $\overline{Q}$ the grids $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau$, $\overline{\omega}_{\alpha h\tau} = \overline{\omega}_{\alpha h} \times \omega_\tau$, $\alpha = 1, 2$, where

$$\overline{\omega}_h = \Big\{x_{i_1 i_2} = (i_1 h_1, i_2 h_2),\ i_\beta = 0, \dots, M_\beta, M_\beta h_\beta = 1,\ \beta = 1,2\Big\},$$
$$\overline{\omega}_{1h} = \Big\{x_{i_1 i_2} = ((i_1 - 1/2)h_1, i_2 h_2),\ i_1 = 1, \dots, M_1,\ i_2 = 0, \dots, M_2\Big\},$$
$$\overline{\omega}_{2h} = \Big\{x_{i_1 i_2} = (i_1 h_1, (i_2 - 1/2)h_2),\ i_1 = 0, \dots, M_1,\ i_2 = 1, \dots, M_2\Big\},$$
$$\omega_h = \Omega \cap \overline{\omega}_h, \quad \gamma_h = \overline{\omega}_h \setminus \omega_h, \quad \overline{\omega}_h = \omega_h \cup \gamma_h, \quad \omega_\tau = \big\{t_j = j\tau,\ j = 0, \dots, N,\ N\tau = T\big\}.$$

Here h_α is the space step in direction x_α and τ is the time step on $[0, T]$.

Define the following inner products and the norms for the discrete functions y and z given on $\overline{\omega}_h$

$$(y,z) = \sum_{i_1=1}^{M_1-1}\sum_{i_2=1}^{M_2-1} y_{i_1 i_2} z_{i_1 i_2} h_1 h_2, \quad (y,z]_1 = \sum_{i_1=1}^{M_1}\sum_{i_2=1}^{M_2-1} y_{i_1 i_2} z_{i_1 i_2} h_1 h_2,$$
$$(y,z]_2 = \sum_{i_1=1}^{M_1-1}\sum_{i_2=1}^{M_2} y_{i_1 i_2} z_{i_1 i_2} h_1 h_2, \quad \|y\| = (y,y)^{1/2}, \quad \|y]|_1 = (y,y]_1^{1/2}, \quad \|y]|_2 = (y,y]_2^{1/2}.$$

The inner products and the norms on $\overline{\omega}_{\alpha h}$ are defined in a similar way.

Using known notations let us correspond to the problem (4)–(7) the difference scheme of the type of variable directions:

$$u_{1t} = (\widehat{v}_1 \widehat{u}_{1\overline{x}_1})_{x_1} + (v_2 u_{2\overline{x}_2})_{x_2}, \quad u_{2t} = (\widehat{v}_1 \widehat{u}_{1\overline{x}_1})_{x_1} + (\widehat{v}_2 \widehat{u}_{2\overline{x}_2})_{x_2}, \tag{11}$$
$$v_{\alpha t} = -\widehat{v}_\alpha + g_\alpha(v_\alpha u_{\alpha \overline{x}_\alpha}), \tag{12}$$
$$u_\alpha(x,0) = U_0(x), \quad x \in \overline{\omega}_h, \quad v_\alpha(x,0) = V_{\alpha 0}(x), \quad x \in \overline{\omega}_{\alpha h}, \tag{13}$$
$$u_\alpha(x,t) = 0, \quad (x,t) \in \gamma_h \times \omega_\tau, \quad \alpha = 1,2. \tag{14}$$

In (11), (12) the discrete functions u_1, u_2 are defined on $\overline{\omega}_{h\tau}$ and v_α on $\overline{\omega}_{\alpha h\tau}$.

Theorem 1. *If the differential problem* (4)–(7) *has the sufficiently smooth solution* U, V_1, V_2, *then there exist* $\tau_0 > 0$ *such that for all* $\tau < \tau_0$ *the scheme* (11)–(14) *is absolutely stable with respect to initial data and the following estimates hold*

$$\|u_{1\overline{x}_1}]|_1^2 + \|u_{2\overline{x}_2}]|_2^2 \le e^{CT}\Big\{\big\|(V_{10}U_{0\overline{x}_1})_{x_1}\big\|^2 + \|U_{0\overline{x}_1}]|_1^2 + \big\|(V_{20}U_{0\overline{x}_2})_{x_2}\big\|^2 + \|U_{0\overline{x}_2}]|_2^2\Big\},$$
$$0 < c \le v_\alpha(x,t) \le C, \quad (x,t) \in \overline{\omega}_{\alpha h\tau}, \quad \alpha = 1,2.$$

Theorem 2. *If the differential problem* (4)–(7) *has the sufficiently smooth solution* U, V_1, V_2, *then the solution of the scheme* (11)–(14) *converges to the exact solution of problem* (4)–(7) *as* $\tau \to 0$, $h_1 \to 0$, $h_2 \to 0$, *and the following inequality holds*

$$\|z_{1\overline{x}_1}]|_1 + \|z_{2\overline{x}_2}]|_2 + \|s_1]|_1 + \|s_2]|_2 \le C(\tau + h_1^2 + h_2^2).$$

Here $z_\alpha = u_\alpha - U$, $s_\alpha = v_\alpha - V_\alpha$, $\alpha = 1, 2$. The statements analogous to Theorems 1 and 2 are true for multi-dimensional (1)–(3) case as well.

On each segment $\Delta_k = [k\tau, (k+1)\tau]$, $k = 1, 2, \dots, N$ for the system (1)–(3) with (6), (7) type initial-boundary conditions let us consider the following averaged model of sum approximation:

$$\eta_i \frac{\partial u_i^k}{\partial t} = \frac{\partial}{\partial x_i}\Big(v_i^k \frac{\partial u_i^k}{\partial x_i}\Big), \quad \frac{\partial v_i^k}{\partial t} = -v_i^k + g_i\Big(v_i^k \frac{\partial u_i^k}{\partial x_i}\Big), \tag{15}$$

$$u_i^k\big|_{x_i=0} = u_i^k\big|_{x_i=1} = 0, \quad u_i^0(x,0) = U_0(x), \quad v_i^0(x,0) = V_{i,0}(x), \tag{16}$$

$$u_i^k(x, t_k) = u^{k-1}(x, t_k), \quad v_i^k(x, t_k) = v_i^{k-1}(x,\, t_k), \tag{17}$$

$$u^k(x,t) = \sum_{i=1}^{p} \eta_i u_i^k(x,t), \quad \eta_i > 0, \quad \sum_{i=1}^{p} \eta_i = 1. \tag{18}$$

Theorem 3. *If the differential problem has the sufficiently smooth solution, then the solution of the averaged model* (15)–(18) *converges to the exact solution as* $\tau \to 0$ *and the following estimate holds*

$$\big\|u^k(t) - U(t)\big\| + \sum_{i=1}^{p} \big\|v_i^k(t) - V_i(t)\big\| = O(\tau^{1/4}).$$

Numerous numerical computations are carried out for two-dimensional biological problem (4)–(7) using (11)–(14) and (15)–(18) models. Numerical experiments agree with theoretical researches.

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