Lipschitz Property of the Lower Sigma-Exponent of Linear Differential System

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Consider the linear differential systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \geqslant 0, \tag{1_A}$$

with bounded piecewise continuous coefficients on the half-line $[0, +\infty)$, with the Cauchy matrix $X_A(t, \tau)$ and with characteristic exponents $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$.

Let $\lambda[Q]$ denote Lyapunov exponent of piecewice continuous nth-order matrix $Q(t)$. For the higher sigma-exponent [1; see also 2; 3, p. 225]

$$\nabla_\sigma(A) \equiv \sup_{\lambda[Q] \leqslant -\sigma} \lambda_n(A+Q), \ \ \sigma > 0, \tag{2}$$

of the linear system (1_A) the algorithm for its computation on the basis of the sequence $\{\xi_k(\sigma)\}$ was suggested in the above-mentioned papers:

$$\nabla_\sigma(A) = \overline{\lim_{k\to\infty}} \frac{\xi_k(\sigma)}{k}, \ \ \sigma > 0. \tag{3_1}$$

The elements of this sequence are defined recursively on the basis of the Cauchy matrix $X_A(t, \tau)$ of the system (1_A):

$$\xi_k(\sigma) = \max_{i<k} \big\{ \ln \|X_A q(k, i)\| + \xi_i(\sigma) - \sigma i \big\}, \ \ \xi_0(\sigma) = 0. \tag{3_2}$$

The higher sigma-exponent $\nabla_\sigma(A)$ as a function of the parameter $\sigma > 0$ is bounded [4, p. 31] non-increasing on the half-line $[0, +\infty)$ and identically equal to [5] the highest characteristic exponent $\lambda_n(A)$ of the system (1_A) on the interval $(\sigma_0, +\infty)$, where $\sigma_0 = \sigma_0(A) \geqslant 0$ is the Grobman irregularity coefficient of the system (1_A). It was shown in [1] that the higher sigma-exponent $\nabla_\sigma(A)$ is continuous function in parameter σ and, moreover, it has the Lipschits property with Lipschitz constant $L(\varepsilon) > 0$ in any interval $[\varepsilon, +\infty)$, $\varepsilon > 0$.

The analysis of properties of the higher sigma-exponent $\nabla_\sigma(A)$ as a function of the parameter $\sigma > 0$ in [6, 7] led to the following its main property–concavity. Finally it was proved [8] on the basis of the algorithm (3_1)–(3_2) that the properties 1) boundedness; 2) concavity; 3) coincidence with a constant for σ exceeding some $\sigma_0 \geqslant 0$ completely determine $\nabla_\sigma(A)$. In particular, for any function $f : (0, +\infty) \to R^1$ with the properties 1)–3) it was proved the existence of the linear system such that its higher sigma-exponent coincides with this function $f(\sigma)$ for all $\sigma > 0$.

As applications of the higher sigma-exponent we will indicate its use in the investigation of the exponential stability of the linear perturbed systems (1_{A+Q}) with exponentially decreasing perturbations Q and nonlinear systems with a linear approximation (1_A) and perturbations of the higher order of smallness.

The lower sigma-exponent [9,10]

$$\Delta_\sigma(A) \equiv \inf_{\lambda[Q] \leqslant -\sigma} \lambda_1(A+Q), \ \ \sigma > 0, \tag{4}$$

is defined by the lowest characteristic exponents $\lambda_1(A+Q)$ of the perturbed linear systems (1_{A+Q}). It is used in the investigation of both the instability of the zero solution $(\Delta_\sigma(A) > 0)$ and in any case the one-dimensional conditional stabilization ($\Delta_\sigma(A) < 0$) of the linear differential system (1_{A+Q}) with the exponentially decreasing perturbation Q. It is also possible to use it in the investigation of the analogous properties of the zero solution of nonlinear differential systems with perturbations of the higher order of smallness in a neighborhood of the origin of coordinates.

For the lower sigma-exponent it is absent any algorithm for its computation on the basis of the Cauchy matrix $X_A(t,\tau)$ of linear system or of its solutions. At the same time such algorithm has been constructed [9; 3, p. 52] for its limit value $\Delta_0(A) \equiv \Delta(A)$ of this exponent when $\sigma \to +0$. From definition (4) of the lower sigma-exponent $\Delta_\sigma(A)$ of the system (1_A) as a function of the parameter $\sigma > 0$ we have the following its properties: 1) boundedness on the half-line $(0,+\infty)$ by virtue of boundedness of the matrix $A(t) + Q(t)$ of coefficients of the perturbed linear system (1_{A+Q}) (see [4, p. 31]); 2) nondecreasing on this half-line by virtue of the restriction of the range of matrix Q when σ increases, $\sigma > 0$, 3) identical coincidence of the exponent $\Delta_\sigma(A)$ with the lowest characteristic exponent $\lambda_1(A)$ of the initial system (1_A) for all $\sigma > \sigma_0(A)$ on the basis of the property of the Grobman irregularity coefficient $\sigma_0(A)$. However, even such fundamental property of continuity or, on the contrary, discontinuity of the lower sigma-exponent $\Delta_\sigma(A)$ as a function of $\sigma \in (0,+\infty)$ is not established in general case. Hence the problem [10] about complete description of the properties of this exponent is unsolved.

In our opinion, the sufficiently significant advance in the investigation of properties of the lower sigma-exponent is the proof of the existense of Lipschitz in $\sigma \in (0,+\infty)$ exponents $\Delta_\sigma(A)$ which has been obtained in [11]. It is self-evident that they have the mentioned above necessary properties 1)–3). At first for this purpose we construct piecewise linear lower sigma-exponents and the corresponding linear differential systems.

Lemma ([11])**.** *For any function*

$$\varphi(\sigma) = \begin{cases} \Delta_0 + (\Delta_1 - \Delta_0)\,\dfrac{\sigma}{\sigma_0} & \text{for } \sigma \in (0,\sigma_0], \\ \Delta_1 & \text{for } \sigma \in (\sigma_0,+\infty), \end{cases}$$

with arbitrary parameters

$$\Delta_1 - \Delta_0 > \sigma_0 > 0,$$

there exists a two-dimensional system (1_A) *with bounded piecewise constant on the half-line* $[0,+\infty)$ *coefficients and with lower sigma-exponent* $\Delta_\sigma(A) \equiv \varphi(\sigma)$, $\sigma \in (0,+\infty)$.

With the help of this lemma and stages of its proof we establish the validity of the necessary assertion.

Theorem 1 ([11])**.** *For any nondecreasing function* $f : (0,+\infty) \to [c_0, d_0] \subset (-\infty + \infty)$ *that coincides with a constant* d_0 *on some interval* $(\sigma_0,+\infty)$ *and satisfies the Lipschitz condition*

$$0 \leqslant f(\sigma_2) - f(\sigma_1) < L(\sigma_2 - \sigma_1), \quad 0 < \sigma_1 < \sigma_2 < +\infty,$$

with a finite Lipschitz constant $L > 1$, *there exists a linear system* (1_A) *with bounded piecewise constant coefficients such that its lower sigma-exponent* $\Delta_\sigma(A)$ *coincides with the function* $f(\sigma)$ *for all* $\sigma > 0$.

In connection with the assertion of this theorem the following question is interesting: will the lower sigma-exponent $\Delta_\sigma(A)$ be Lipschitz function in the whole range of definition $(0,+\infty)$? The negative answer to it is given by

Theorem 2. *For any nondecreasing bounded on the half-line* $[0,+\infty)$ *function* $f(\sigma)$ *coincident with a constant* d_0 *on some interval* $(\sigma_0,+\infty)$, $\sigma_0 > 0$, *and satisfying on any interval* $[\varepsilon,+\infty) \subset (0,+\infty)$ *the Lipschitz condition with Lipschitz constant* $L(\varepsilon) < +\infty$, $\varepsilon > 0$, *there exists a linear system* (1_A) *with bounded piecewise continuous coefficients and the lower sigma-exponent* $\Delta_\sigma(A) \equiv f(\sigma)$, $\sigma > 0$.

Remark. Evidently there is a function $f(\sigma)$ satisfying all the conditions of Theorem 2 and not satisfying the Lipschitz condition with finite Lipschitz constant on the whole interval.

We note that at the investigation of lower sigma-exponents the following problems remains unsolved.

Problem 1. *To determine whether the sufficient properties of the lower sigma-exponent obtained in Theorem* 2 *are also necessery.*

Problem 2. *To construct some algorithm of the calculation of the lower sigma-exponent of the system* (1_A) *on the basis of its Cauchy matrix or of its solutions.*

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