On Some Properties of Irreducibility Sets of Linear Differential Systems

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We consider linear systems of the form

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \in I = [0, +\infty), \tag{1}$$

with piecewise continuous bounded coefficients ($\|A(t)\| \le a$ for $t \in I$). Along with original systems (1) we will consider the perturbed systems

$$\dot{y} = (A(t) + Q(t))y, \;\; y \in \mathbb{R}^n, \;\; t \in I, \tag{2}$$

with piecewise continuous perturbations Q defined on I and satisfying either the condition

$$\|Q(t)\| \le C_Q e^{-\sigma t}, \;\; \sigma \ge 0, \;\; t \in I, \tag{3}$$

or more general condition

$$\|Q(t)\| \le C_Q^{\varepsilon} e^{(\varepsilon-\sigma)t}, \;\; \sigma \ge 0, \;\; \forall\, \varepsilon > 0, \;\; t \in I, \tag{4}$$

which is equivalent to the inequality $\lambda[Q] \equiv \overline{\lim\limits_{t\to+\infty}}\, t^{-1} \ln \|Q(t)\| \le -\sigma \le 0$.

If $\sigma = 0$ in (3), (4), then we additionally suppose that $Q(t) \to 0$ as $t \to +\infty$.

Following Yu. S. Bogdanov [1] we say that systems (1) and (2) are asymptotically equivalent (Lyapunov's equivalent, reducible) if there exists a Lyapunov transformation

$$x = L(t)y, \quad \max\Big\{\sup_{t\in I} \|L(t)\|, \sup_{t\in I} \|L^{-1}(t)\|, \sup_{t\in I} \|\dot{L}(t)\|\Big\} < +\infty,$$

reducing one of them to the other.

The sets $N_r(a, \sigma)$, $N_\rho(a, \sigma)$, $a > 0$, $\sigma \ge 0$, are said to be the irreducibility sets if they consist of all systems (1) with the following properties [2]:

(1) the norm of the coefficient matrix A is less than or equal to a on I;

(2) for each system (1) there exists a system (2) with the matrix Q satisfying either the condition (3) or the more general condition (4), respectively, which cannot be reduced to system (1).

If Q satisfies (3) or (4) with $\sigma > 2a$, then $\Big\| \int\limits_t^{+\infty} Q(u)\, du \Big\| \le Ce^{-\sigma_1 t}$ for some $C > 0$ and $\sigma_1 > 2a$, so [3] systems (1) and (2) are asymptotically equivalent, and, therefore, the sets $N_r(a, \sigma)$, $N_\rho(a, \sigma)$ are empty for all $\sigma > 2a$.

From the properties of perturbation matrix Q it follows that $N_r(a, 0) = N_\rho(a, 0)$ for all $a > 0$, and these sets are nonempty and related by the inclusion $N_r(a, \sigma) \subseteq N_\rho(a, \sigma)$ for $\sigma \in (0, 2a]$. Moreover [4], $N_r(a, \sigma) \ne N_\rho(a, \sigma)$ for $\sigma \in (0, 2a]$.

It is obvious that $N_r(a,\sigma_2) \subseteq N_r(a,\sigma_1)$, $N_\rho(a,\sigma_2) \subseteq N_\rho(a,\sigma_1)$ for $0 < \sigma_1 < \sigma_2 \le 2a$, and [4] these irreducibility sets strictly decrease with the increasing parameter σ, i.e.

$$N_r(a,\sigma_2) \subset N_r(a,\sigma_1), \quad N_\rho(a,\sigma_2) \subset N_\rho(a,\sigma_1) \ \ \forall\, 0 < \sigma_1 < \sigma_2 \le 2a.$$

We introduced [2, 5] the reducibility coefficient (the reducibility exponent) of the linear system (1) as the greatest lower bound of the set of all values of the parameter $\sigma > 0$ for which the perturbed system (2) with an arbitrary perturbation Q satisfying condition (3) (respectively, condition (4)) can be reduced to the original system (1). The properties of these reducibility coefficients and exponents allows us to investigate certain properties of the irreducibility sets as the functions of the parameters a and σ. The following statements are hold.

Theorem 1. *For all* $a_0 > 0$, $\sigma \in (0, 2a_0)$ *we have*

$$\operatorname*{Lim}_{a\to a_0-0} N_r(a,\sigma) \neq N_r(a_0,\sigma), \quad \operatorname*{Lim}_{a\to a_0+0} N_r(a,\sigma) = N_r(a_0,\sigma),$$
$$\operatorname*{Lim}_{a\to a_0-0} N_\rho(a,\sigma) \neq N_\rho(a_0,\sigma), \quad \operatorname*{Lim}_{a\to a_0+0} N_\rho(a,\sigma) = N_\rho(a_0,\sigma).$$

Theorem 2. *For all* $a > 0$, $\sigma_0 \in (0, 2a]$ *we have*

$$\operatorname*{Lim}_{\sigma\to \sigma_0-0} N_r(a,\sigma) \neq N_r(a,\sigma_0), \quad \operatorname*{Lim}_{\sigma\to \sigma_0+0} N_r(a,\sigma) \neq N_r(a,\sigma_0).$$

Theorem 3. *For all* $a > 0$, $\sigma_0 \in (0, 2a]$ *we have*

$$\operatorname*{Lim}_{\sigma\to \sigma_0-0} N_\rho(a,\sigma) = N_\rho(a,\sigma_0), \quad \operatorname*{Lim}_{\sigma\to \sigma_0+0} N_\rho(a,\sigma) \neq N_\rho(a,\sigma_0).$$

References

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