Variation Formulas of Solution for a Nonlinear Functional Differential Equation Taking into Account Two Delay Parameters Perturbation and the Continuous Initial Condition

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Let $I = [a, b]$ be a finite interval and let $0 < \tau_1 < \tau_2$, $0 < \sigma_1 < \sigma_2$ be given numbers; suppose that $O \subset \mathbb{R}^n$ is an open set and E is a set of n-dimensional functions $f(t, x, y, z)$ satisfying the following conditions: for almost all $t \in I$ the function $f(t, \cdot) : O^3 \to \mathbb{R}^n$ is continuously differentiable; for any fixed $(x, y, z) \in O^3$ the functions $f(t, x, y, z)$, $f_x(\cdot)$, $f_y(\cdot)$, $f_z(\cdot)$ are measurable on I; for each $f \in E$ and compact set $K \subset O$ there exists a function $m_{f,K}(t) \in L(I, [0, \infty))$ such that

$$|f(t, x, y, z)| + |f_x(t, x, y, z)| + |f_y(t, x, y, z)| + |f_z(t, x, y, z)| \leq m_{f,K}(t)$$

for all $(x, y, z) \in K^3$ and for almost all $t \in I$. Further, let Φ be the set of continuous functions $\varphi : I_1 \to O$, where $I_1 = [\widehat{\tau}, b]$, $\widehat{\tau} = a - \max\{\tau_2, \sigma_2\}$.

To each element $\mu = (t_0, \tau, \sigma, \varphi, f) \in \Lambda = [a, b) \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times \Phi \times E$ we assign the functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t - \tau), x(t - \sigma)\big), \ \ t \in [t_0, t_1] \tag{1}$$

with the continuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0]. \tag{2}$$

Definition. Let $w = (t_0, \tau, \sigma, \varphi, f) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_0 < t_1 \leq b$, is called the solution of equation (1) with the initial condition (2) or the solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$, if $x(t)$ satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let us introduce the set of variations

$$V = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta\sigma, \delta\varphi, \delta f) : \ |\delta t_0| \leq \alpha, \ |\delta\tau| \leq \alpha, \ |\delta\sigma| \leq \alpha, \\ \delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \ |\lambda_i| \leq \alpha, \ i = \overline{1, k}\Big\},$$

where $\delta\varphi_i \in \Phi - \varphi_0$ $\delta f_i \in E - f_0$, $i = \overline{1, k}$, $\varphi_0 \in \Phi$, $f_0 \in E$ are fixed functions, $\alpha > 0$ is a fixed number. Let $x_0(t)$ be the solution corresponding to the element $\mu_0 = (t_{00}, \tau_0, \sigma_0, \varphi_0, f_0) \in \Lambda$ with $t_{00} \in (a, b), \tau_0 \in (\tau_1, \tau_2), \sigma_0 \in (\sigma_1, \sigma_2)$ and defined on the interval $[\widehat{\tau}, t_{10}], t_{10} \in (t_{00}, b)$. There exist numbers $\delta_1 > 0$ and $\varepsilon > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$ we have $\mu_0 + \varepsilon\delta\mu \in V$ and to the element $\mu_0 + \varepsilon\delta\mu$ there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$.

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Let us define the increment of the solution $x_0(t) := x(t; \mu_0)$:

$$\delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \ \ \forall\, (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times [0, \varepsilon_1] \times V.$$

Theorem 1. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous and let the functions $\dot\varphi_0(t)$ and $f_0(t,x,y,z),(t,x,y,z)\in I\times O^3$ be bounded. Moreover, there exist the finite limits*

$$\dot\varphi_0^- = \lim_{t\to t_{00}-}\dot\varphi_0(t), \quad \lim_{w\to w_0} f_0(w)=f_0^-, \quad w=(t,x,y,z)\in(a,t_{00}]\times O^3,$$

where $w_0=(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0),\varphi_0(t_{00}-\sigma_0))$. Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that

$$\delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\delta\mu)+o(t;\varepsilon\delta\mu) \tag{3}$$

for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{00},t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^-$, where $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$ and

$$\delta x(t;\delta\mu)=Y(t_{00};t)(\dot\varphi_0^- - f_0^-)\delta t_0+\beta(t;\delta\mu),$$

$$\beta(t;\delta\mu)=Y(t_{00};t)\delta\varphi(t_{00})+\int\limits_{t_{00}-\tau_0}^{t_{00}} Y(s+\tau_0;t)f_{0y}[s+\tau_0]\delta\varphi(s)\,ds+$$

$$+\int\limits_{t_{00}-\sigma_0}^{t_{00}} Y(s+\sigma_0;t)f_{0z}[s+\sigma_0]\delta\varphi(s)\,ds-\Bigg[\int\limits_{t_{00}}^{t} Y(s;t)f_{0y}[s]\dot x_0(s)\,ds\Bigg]\delta\tau-$$

$$-\Bigg[\int\limits_{t_{00}}^{t} Y(s;t)f_{0z}[s]\dot x_0(s)\,ds\Bigg]\delta\sigma+\int\limits_{t_{00}}^{t} Y(s;t)\delta f[s]\,ds.$$

Here $Y(s;t)$ is the $n\times n$-matrix function satisfying the equation

$$Y_s(s;t)=-Y(s;t)f_{0x}[s]-Y(s+\tau_0;t)f_{0y}[s+\tau_0]-Y(s+\sigma_0;t)f_{0z}[s+\sigma_0],\quad s\in[t_{00},t]$$

and the condition $Y(s;t)=\begin{cases}H, & s=t,\\ \Theta, & s>t,\end{cases}$ where H is the identity matrix,Θ is the zero matrix;

$$f_{0y}[t]=f_{0y}\big(t,x_0(t),x_0(t-\tau_0),x_0(t-\sigma)\big),\quad \delta f[t]=\delta f\big(t,x_0(t),x_0(t-\tau_0),x_0(t-\sigma)\big).$$

The Theorem 1 is proved by the scheme described in [1].

Theorem 2. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous and let the functions $\dot\varphi_0(t)$ and $f_0(t,x,y,z),(t,x,y,z)\in I\times O^3$ be bounded. Moreover, there exist the finite limits*

$$\dot\varphi_0^+ = \lim_{t\to t_{00}+}\dot\varphi_0(t), \quad \lim_{w\to w_0} f_0(w)=f_0^+, \quad w\in[t_{00},t_{10}]\times O^3.$$

Then for each $\widehat t_0\in(t_{00},t_{10})$ there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for any $(t,\varepsilon,\delta\mu)\in[\widehat t_0,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^+$, where $V^+=\{\delta\mu\in V:\delta t_0\ge 0\}$, formula (3) *holds, where $\delta x(t;\delta\mu)=Y(t_{00};t)(\dot\varphi_0^+ - f_0^+)\delta t_0+\beta(t;\delta\mu)$.*

Theorem 3. *Let the assumptions of Theorems* 1 *and* 2 *be fulfilled. Moreover, $\dot\varphi_0^- - f_0^- = \dot\varphi_0^+ - f_0^+ := \widehat f_0$. Then for each $\widehat t_0\in(t_{00},t_{10})$ there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for any $(t,\varepsilon,\delta\mu)\in[\widehat t_0,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V$, formula* (3) *holds, where $\delta x(t;\delta\mu)=Y(t_{00};t)\widehat f_0\delta t_0+\beta(t;\delta\mu)$.*

All assumptions of Theorem 3 are satisfied if the function $f_0(t,x,y,z)$ is continuous and bounded, and the function $\varphi_0(t)$ is continuously differentiable. It is clear that in this case

$$\widehat f_0=\dot\varphi_0(t_{00})-f_0\big(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0),\varphi_0(t_{00}-\sigma_0)\big).$$

References

[1] T. Tadumadze, Variation formulas of solution for a delay differential equation taking into account delay perturbation and the continuous initial condition. *Georgian Math. J.* **18** (2011), No. 2, 345–364.