Existence of 2π-Periodic Solutions for the Brillouin Electron Beam Focusing Equation

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1 Introduction

This note concerns the problem of existence of periodic solutions of the equation

$$\ddot{x} + b(1 + \cos t)x = \frac{1}{x}\,, \tag{1}$$

where b is a positive constant. Throughout the paper, we will not take into account solutions with collisions, but we will always search for positive 2π-periodic solutions of (1).

The physical meaning of equation (1) arises in the context of Electronics, since it governs the motion of a magnetically focused axially symmetric electron beam under the influence of the Brillouin flow, as shown in [1]. From a mathematical point of view, (1) is a singular perturbation of a Mathieu equation.

Motivated by some numerical experiments realized in [1], where it was conjectured that, if $b \in (0, 1/4)$, equation (1) should have a 2π-periodic solution, in the last fifty years the work of many mathematicians has given birth to an extensive literature about this topic. Although at the moment the conjecture has not been correctly proven yet, many advances in this line have been obtained, allowing to understand that the problem of existence of 2π-periodic solutions of (1) when $b \in (0, 1/4)$ can be really delicate, and arising doubts on the validity of the result conjectured in [1].

To the best of our knowledge, M. Zhang determined in [7] the best range of b actually known for the solvability of (1), using a non-resonance hypothesis for the associated Mathieu equation. He proves that whether $b \in (0, 0.16448)$, then (1) has at least one 2π-periodic solution. This last result has been extended to equations where the singularity may be of weak type (see [6]).

An important result to understand the difficulty of showing the validity of the conjecture proposed in [1] was proven in [8]. In that paper, it was established an unanimous relation between the stability intervals for the Mathieu equation and the existence of periodic solutions for the Yermakov–Pinney equation. Notice that the stability intervals of the Mathieu equation $(\lambda_0, \lambda_1'), (\lambda_2', \lambda_1), (\lambda_2, \lambda_3'), \dots,$ are defined approximately by $\lambda = 0,\ \lambda_1' \approx 1/6;\ \lambda_2' \approx 0.4,\ \lambda_1 \approx 0.95, \dots$ (see [5, Theorem 2.1] and [2, Figure 1]). This suggests that, in order to obtain a correct proof of the conjecture by V. Bevc, J. L. Palmer and C. Süsskind, one has to take into account some property of equation (1) which is not verified for the Yermakov–Pinney equation.

Quite unexpectedly with respect to the numerical and analytical results found in literature, we establish a new range for the existence of $2\pi-$periodic solutions of the Brillouin focusing beam equation. This is possible thanks to suitable nonresonance conditions acting on the rotation number of the solutions in the phase plane. Our main result is

Theorem 1. *If* $b \in [0.4705, 0.59165]$*, then* (1) *has at least one* 2π*-periodic solution.*

2 A Non-Resonance Theorem for Singular Perturbations of a Mathieu Equation

The proof of Theorem 1 is based on a non-resonance result which involves nonlinearities with "atypical" linear growing type, and could have interest by itself.

Theorem 2. *Let us assume that there exist positive constants A_+, B_+ such that*

$$\frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)}{B_+}, 1\right\} dt > \frac{n}{2\sqrt{B_+}}\,, \tag{2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)}{A_+}, 1\right\} dt < \frac{n+1}{2\sqrt{A_+}}\,, \tag{3}$$

for some natural number n. Then (1) *has at least one 2π-periodic solution.*

A couple of remarks are in order.

Remark 1. With the aim of keeping the exposition at a rather simple level, and taking into account that our main goal will be to study the existence of 2π-periodic solutions of (1), we will always consider equation (1) as a starting point. However, the result can be extended, with the same approach and similar computations, to more general equations like

$$\ddot{x} + q(t)x - g(t, x) = 0, \tag{4}$$

where q is continuous and 2π-periodic and positive on almost interval $[0, 2\pi]$, and $g : [0, 2\pi] \times (0, +\infty) \to \mathbb{R}$ has a similar behavior as $1/x^\gamma$, with $\gamma \geq 1$, being allowed to grow at most sublinearly at infinity. Of course, in this case $q(t)$ will replace $b(1 + \cos t)$.

Remark 2. Conditions (2) and (3) were introduced by Fabry in [3] for the equation

$$\ddot{x} + g(t, x) = 0,$$

with

$$p(t) \leq \liminf_{|x|\to+\infty} \frac{g(t, x)}{x} \leq \limsup_{|x|\to+\infty} \frac{g(t, x)}{x} \leq q(t),$$

asking that

$$\sqrt{\lambda_j} < \sup_{\xi>0} \frac{\frac{1}{2\pi}\int_0^{2\pi} \min\{p(t), \xi\}\, dt}{\sqrt{\xi}}\,, \qquad \inf_{\xi>0} \frac{\frac{1}{2\pi}\int_0^{2\pi} \max\{q(t), \xi\}\, dt}{\sqrt{\xi}} < \sqrt{\lambda_{j+1}}\,,$$

where λ_j is the j-th eigenvalue of the considered 2π-periodic problem. Such conditions are usually coupled with the sign assumption $\liminf_{|x|\to+\infty} \operatorname{sgn} x f(t, x) > 0$, which, however, in our model, is not satisfied.

As it is easy to see, (2) and (3) are the counterpart of such conditions for the Dirichlet spectrum (which is the natural one to consider when dealing with problems with a singularity).

3 Application

In order to prove Theorem 1 it will be convenient, for any $n \in \mathbb{N}$, to define the absolutely continuous functions $F_n, G_n : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$ by

$$F_n(b, x) = \frac{1}{2\pi}\int_0^{2\pi} \min\left\{\frac{b(1+\cos t)}{\sqrt{x}}\,, \sqrt{x}\right\} dt - \frac{n}{2}\,,$$

$$G_n(b, x) = \frac{1}{2\pi}\int_0^{2\pi} \max\left\{\frac{b(1+\cos t)}{\sqrt{x}}\,, \sqrt{x}\right\} dt - \frac{n+1}{2}\,.$$

Both functions are non-decreasing with respect to the variable b. Moreover, if there exists $n \in \mathbb{N}$ such that $\inf_{x>0} G_n(b,x) < 0$ and $\sup_{x>0} F_n(b,x) > 0$, then Theorem 2 implies that (1) has at least one periodic solution. Therefore, we have the following proposition.

Proposition 1. *Assume that there exists $n \in \mathbb{N}$ such that*

$$b \in \left(\inf \left\{ b > 0 : \sup_{x>0} F_n(b,x) > 0 \right\}, \sup \left\{ b > 0 : \inf_{x>0} G_n(b,x) < 0 \right\} \right). \tag{5}$$

Then (1) *has at least one 2π-periodic solution.*

Let us first observe that, in view of the continuity and the monotonicity of the functions F_n, G_n in the variable b, there exist b_0^n and b_1^n such that

$$\left\{ b > 0 : \sup_{x>0} F_n(b,x) > 0 \right\} = (b_0^n, +\infty) \text{ and } \left\{ b > 0 : \inf_{x>0} G_n(b,x) < 0 \right\} = (0, b_1^n).$$

The point is to prove that these two intervals contain common points, i.e., $b_0^n < b_1^n$. We will show this in the case when $n = 0$ and $n = 1$, and the estimates performed in this last case will allow to achieve the new result consisting in Theorem 1.

Remark 3. To see the details of the proofs, this results are published in [4].

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