Asymptotic Behaviour of Solutions of Nonautonomous Ordinary Differential Equations with Rapidly Varying Nonlinearities

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The monograph by I. T. Kiguradze, T. A. Chanturiya [1] summarizes numerous studies of asymptotic properties of solutions of nonautonomous differential equations by 1990. In particular, the asymptotic behavior of solutions of binomial differential equations with power nonlinearities (Equations of Emden-Fowler type) is sufficiently well described there.

The following differential equation

$$y^{(n)} = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ – continuous function, $-\infty < a < \omega \leq +\infty$, $\varphi : \Delta_{Y_0} \to]0, +\infty[$ – twice continuously differentiable function satisfying the conditions

$$\varphi'(y) \neq 0 \quad \text{if } \ y \in \Delta_{Y_0}, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{or} & 0, \\ \text{or} & +\infty, \end{cases} \tag{2}$$

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{[\varphi'(y)]^2}{\varphi''(y)\varphi(y)} = 1, \tag{3}$$

Y_0 is zero, or $\pm\infty$, Δ_{Y_0}– one-sided neighborhood of Y_0, is considered in this paper.

By (2) and (3), the function $\varphi''(y)$ is nonzero in some neighborhood of Y_0, that is contained in Δ_{Y_0}. For definiteness, without loss of generality, we assume that

$$\Delta_{Y_0} = \begin{cases} [y_0, Y_0[\,, & \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0, \\ ]Y_0, y_0], & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0, \end{cases}$$

where $y_0 \in \mathbf{R}$ satisfies the inequality $|y_0| > 1$ ($|y_0| < 1$), if $Y_0 = \pm\infty$ ($Y_0 = 0$) and $\varphi''(y) \neq 0$, if $y \in \Delta_{Y_0}$.

Moreover, from (2) and (3) it follows that

$$\lim_{y\to Y_0} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty, \quad \lim_{y\to Y_0} \frac{y\varphi''(y)}{\varphi'(y)} = \pm\infty.$$

Hence the functions $\varphi(y)$ and $\varphi'(y)$ are rapidly varying if $y \to Y_0$ in the sense of definition from monograph Bingham N. H., Goldie C. M., Teugels J. L. [2, Ch. 2, § 2.4, p. 83]. Assuming

$$\mu_0 = \operatorname{sign} \varphi'(y) \quad \text{if } \ y \in \Delta_{Y_0},$$

let us notice that $\varphi(y)$ and $\varphi'(y)$ are rapidly tending to zero if $y \to Y_0$ in the cases

$$\mu_0 y_0 > 0, \ \ Y_0 = 0 \ \text{ or } \ \mu_0 y_0 < 0, \ \ Y_0 = \pm\infty,$$

and rapidly tending to infinity in the cases

$$\mu_0 y_0 < 0, \ \ Y_0 = 0 \ \text{ or } \ \mu_0 y_0 > 0, \ \ Y_0 = \pm\infty.$$

In the case $\varphi(y) = e^{\sigma y}$ ($\sigma \neq 0$) and $Y_0 = +\infty$, in works [3, 4] the asymptotic behavior of the solutions of differential equations (1) with rapidly varying functions φ was researched earlier for $n = 2$ and in works [5–7] for $n \geq 2$. The case $n = 2$ was also considered for an arbitrary function $\varphi : \Delta_{Y_0} \to]0, +\infty[$, that satisfies the conditions (2) and (3) in monograph V. Maric [8] and in work [9]. The case $Y_0 = 0$ and $\omega = +\infty$ was researched in [8]. The case $n = 2$, arbitrary $Y_0 \in \{0, \pm\infty\}$ and $\omega \leq +\infty$ was considered in [9]. It should be noted, that in [9] the class of solutions defined after the function φ was studied.

In this work we leave the class of solutions the same as it was researched earlier (for instance in [10]) for equations with regularly varying as $y \to Y_0$ functions φ.

Definition. Solution y of differential equation (1), defined on $[t_0, \omega[\subset \Delta_{Y_0}$, is called $P_\omega(Y_0, \lambda^0_{n-1})$ – solution, where $-\infty \leq \lambda^0_{n-1} \leq +\infty$, if it satisfies the conditions

$$y(t) \in \Delta_{Y_0} \ \text{ if } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y(t) = Y_0,$$

$$\lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty, \end{cases} \qquad (k = \overline{1, n-1}),$$

$$\lim_{t \uparrow \omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t) y^{(n-2)}(t)} = \lambda^0_{n-1}.$$

Let us put the following subsidiary notations.

$$a_{0k} = (n-k)\lambda_0 - (n-k-1) \ \ (k = \overline{1, n}) \ \text{ if } \ \lambda_0 \in \mathbb{R},$$

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t - \omega, & \text{if } \omega < +\infty, \end{cases}, \quad J(t) = \int_A^t \pi_\omega^{n-1}(\tau) p(\tau)\, d\tau, \quad \Phi(y) = \int_B^y \frac{ds}{\varphi(s)},$$

$$q_1(t) = \frac{\prod_{j=1}^{n-1} a_{0j}}{\alpha_0(\lambda^0_{n-1} - 1)^n p(t) \pi_\omega^n(t)} \, \frac{\Phi^{-1}\Big(\frac{\alpha_0(\lambda^0_{n-1} - 1)^{n-1}}{\prod_{j=2}^n a_{0j}} J(t)\Big)}{\varphi\Big(\Phi^{-1}\Big(\frac{\alpha_0(\lambda^0_{n-1} - 1)^{n-1}}{\prod_{j=2}^n a_{0j}} J(t)\Big)\Big)},$$

where the limit of integration $A \in \{a, \omega\}$ ($B \in \{y_0, Y_0\}$) is chosen so that the integral tends to zero or to $\pm\infty$ if $t \uparrow \omega$ ($y \to Y_0$), and Φ^{-1} – the inverse function to Φ.

The following result that is related to the not singular case was found for the differential equation (1).

Theorem. *Let* $\lambda^0_{n-1} \in \mathbf{R} \setminus \{0, \frac{1}{2}, \ldots, \frac{n-2}{n-1}, 1\}$. *Then for the existence of the* $P_\omega(Y_0, \lambda^0_{n-1})$ *– solutions of the differential equation* (1) *it is necessary that*

$$\mu_0 \alpha_0 [\lambda^0_{n-1} - 1]^{n-1} \Big(\prod_{j=2}^n a_{0j} \Big) J(t) < 0, \quad \mu_0 y_0 (\lambda^0_{n-1} - 1) a_{01} \pi_\omega^n(t) J(t) < 0 \quad \text{if } t \in]a, \omega[\,, \tag{4}$$

$$\frac{\alpha_0(\lambda^0_{n-1} - 1)^{n-1}}{\prod_{j=2}^n a_{0j}} \lim_{t \uparrow \omega} J(t) = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \Phi(y), \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) J'(t)}{J(t)} = \pm\infty, \quad \lim_{t \uparrow \omega} q_1(t) = 1. \tag{5}$$

Moreover, for each such solution if $t \uparrow \omega$ the following asymptotic representations take place:

$$\varphi'(y(t)) = -\frac{\alpha_0 \prod_{j=2}^n a_{0j}}{(\lambda^0_{n-1} - 1)^{n-1}} \, \frac{1 + o(1)}{J(t)},$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda^0_{n-1} - 1) \pi_\omega(t)} [1 + o(1)] \ \ (k = \overline{1, n-1}),$$

If for some $\lambda_{n-1}^0 \in \mathbf{R} \setminus \left\{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}, 1\right\}$ *the following condition*

$$\lim_{t\uparrow\omega} [1 - q_1(t)] \ln |\pi_\omega(t)| = 0$$

is observed together with (4)–(5) *and algebraic about* ρ *equation*

$$\sum_{k=0}^{n-3} \prod_{i=k+2}^{n-1} a_{0i} \prod_{i=1}^{k} (a_{0i} + \rho) = (\lambda_{n-1}^0 - 1 - \rho) \prod_{i=1}^{n-2} (a_{0i} + \rho) \tag{6}$$

has no roots with zero real part, then (1) *has at least one* $P_\omega(Y_0, \lambda_{n-1}^0)$ *– solution. If (6) has* m *roots, real parts of which have the sign that is opposite to the sign of the function* $(\lambda_{n-1}^0 - 1)\pi_\omega(t)$ *on* $[t_0, \omega[$ *, then if* $CJ'(t)J(t) > 0$ *the equation* (1) *has* m*-parametric family, and if* $CJ'(t)J(t) < 0$ *–* $m + 1$*-parametric family of such solutions, where* C *is defined from the equation*

$$\frac{1}{C} = 1 + \frac{a_{01}}{a_{02}} + \frac{a_{01}}{a_{03}} + \dots + \frac{a_{01}}{a_{0n-2}} + \frac{a_{01}}{a_{0n-1}} - a_{01}.$$

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