Modeling and Optimal Control of One Commodity Production and Supply System

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If some company produces and supplies its commodities to the marker, the management of the company must carefully keep up with demands of the market. It is clear that overproduction and the necessity to store commodities lead to obvious losses for the company. Losses will be even bigger in the case of shortage, aking into consideration the unmet demand and a smaller profit. It is obvious that the fact that shortage impairs the company's reputation should also be take into account.

Let T be the planning period, $t = 0, 1, 2, \dots, T$, be discrete moments of time of commodity supplies to the market, $\tau(t)$, $t = 0, 1, 2, \dots, T$, be the demand function. It is assumed that $\tau(t)$ is the known function.

Denote by $x(t)$, $t = 0, 1, 2, \dots, T$, the supply volume. If the supply $x(t)$ and the demand $\tau(t)$ do not coincide, then the company bears a loss.

Let us introduce the notation $y(t) = x(t) - \tau(t)$. If $y(t) < 0$, then shortage takes place. The company loses the profit which could have been received if the commodity had been sold in a quantity $y(t)$. If $y(t) > 0$, then losses are caused by the necessity to search for new consumers and by unsold commodity storage.

Depending on practical situations, to evaluate losses the function $f_1(y)$ can be written as follows:

$$f_1(y) = \begin{cases} \varphi_1(y), & y < 0, \\ 0, & y = 0, \\ \varphi_2(y), & y > 0, \end{cases}$$

where φ_1, φ_2 increase when the absolute value $|y|$ grows. We can actually assume that losses caused shortage ($y < 0$) exceed losses incurred in case the supply volume exceeds the demand ($y > 0$). Therefore it can be assumed that the derivatives of the functions φ_1, φ_2 satisfy the condition $\varphi_1'(|y|) > \varphi_2'(|y|)$.

As an example let us consider the function

$$f_1(y) = \begin{cases} a_1 y^2, & y < 0, \\ 0, & y = 0, \quad a_1 > b_1 > 0, \\ b_1 y^2, & y > 0. \end{cases} \tag{1}$$

Let $u(t) = x(t+1) - x(t)$. The situation in which the production level is constant, i.e. $x(t) =$ *const* is the most preferable one. In that case, $u(t) = 0$. In the case of an increase of the production output ($u(t) > 0$) and its decrease ($u(t) < 0$), the producer bears losses because of the necessity to reorganize the production.

We call the function $u(t)$ the change dynamics function of the production volume. By analogy with (1), let us introduce the production loss function $f_2(u)$

$$f_2(u) = \begin{cases} a_2 u^2, & u > 0, \\ 0, & u = 0, \quad a_2, b_2 > 0, \\ b_2 u^2, & u < 0. \end{cases}$$

Which of the values a_2 and b_2 is greater depends on a concrete production situation.

Let us now formulate the discrete problem of optimal control: find a change dynamics function $u(t) \in R$, $t = 0, 1, \dots, T$ such that the total loss during the planning period T take a minimal value i.e.

$$\sum_{t=0}^{T-1} \Big(f_1\big(x(t) - \tau(t)\big) + f_2\big(u(t)\big)\Big) + f_1\big(x(T) - \tau(T)\big) \to \min, \tag{2}$$

where $x(t)$ is solution of the discrete equation

$$x(t+1) - x(t) = u(t), \quad t = 0, 1, \dots, T-1 \tag{3}$$

with the initial condition

$$x(0) = x_0. \tag{4}$$

Let us consider the continuous version of the problem (2)–(4): in the set of piecewise-continuous functions $u(t) \in R$, $t \in [0, T]$, with finitely many discontinuities of the first kind, find a function $u_0(t)$ such that the total loss take a minimal value i.e.,

$$\int_0^T \Big(f_1\big(x(t) - \tau(t)\big) + f_2\big(u(t)\big)\Big)\, dt + f_1\big(x(T) - \tau(T)\big) \to \min,$$

where $x(t)$ is solution of the differential equation

$$\dot{x}(t) = u(t), \quad t \in [0, T]$$

with the initial condition (4).

On the basis of the maximum principle [1], it is obtained that if $(\psi(t), x(t))$ is a solution of the following boundary value problem

$$\dot{\psi}(t) = \begin{cases} 2a_1\big(x(t) - \tau(t)\big), & x(t) \ge \tau(t), \\ 2b_1\big(x(t) - \tau(t)\big), & x(t) \le \tau(t), \end{cases}$$

$$\dot{x}(t) = \begin{cases} \psi(t)/2\, a_2, & \psi(t) \ge 0, \\ \psi(t)/2\, b_2, & \psi(t) \le 0, \end{cases}$$

$$\psi(T) = \begin{cases} -2a_1\big(x(T) - \tau(T)\big), & x(T) \ge \tau(T), \\ -2b_1\big(x(T) - \tau(T)\big), & x(T) \le \tau(T), \end{cases}$$

$$x(0) = x_0,$$

then

$$u_0(t) = \begin{cases} \psi(t)/2\, a_2, & \psi(t) \ge 0, \\ \psi(t)/2\, b_2, & \psi(t) \le 0. \end{cases}$$

References

[1] L. S. Pontryagin, V. G. Boltyanskiĭ, R. V. Gamkrelidze, E. F. Mishchenko, The mathematical theory of optimal processes. (Russian) *"Nauka", Moscow*, 1983.