Nonoscillation and Exponential Stability of Second Order Delay Differential Equations without Damping Term

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Delays, arising in nonoscillatory and stable ordinary differential equations, can induce oscillation and instability of their solutions. That is why the traditional direction in the study of nonoscillation and stability of delay equations is to establish a smallness of delay, allowing delay differential equations to preserve these convenient properties of ordinary differential equations with the same coefficients. In this paper, we find cases in which delays, arising in oscillatory and asymptotically unstable ordinary differential equations, induce nonoscillation and stability of delay equations. We demonstrate that, although the ordinary differential equation

$$x''(t)+c(t)x(t)=0$$

can be oscillatiting and asymptoticaly unstable, the delay equation

$$x''(t)+a(t)x(t-h(t))-b(t)x(t-g(t))=0, \ \text{ where } \ c(t)=a(t)-b(t)$$

can be nonoscillating and exponentially stable.

Let us consider the equation

$$x''(t)+a(t)x(t-\tau(t))-b(t)x(t-\theta(t))=0, \ \ t\in[0,+\infty), \qquad (1)$$
$$x(\xi)=0 \ \text{ for } \ \xi<0,$$

where $a(t)$, $b(t)$, $\tau(t)$ and $\theta(t)$ are measurable essentially bounded nonnegative functions. We denote

$$q_*=\operatorname*{ess\,inf}_{t\geq 0} q(t), \quad q^*=\operatorname*{ess\,sup}_{t\geq 0} q(t).$$

Theorem. *Assume that* $0\leq\tau(t)\leq\theta(t)$, $0\leq b(t)\leq a(t)$,

$$4\big\{a(t)-b(t)\big\}\leq\big[b(\theta-\tau)\big]_*^2, \ \ t\in[0,+\infty),$$
$$0<\big[b(\theta-\tau)\big]^*\theta^*\leq\frac{1}{e}\,. \qquad (2)$$

Then

(1) *the Cauchy function* $C(t,s)$ *of equation* (1) *is positive for* $0\leq s<t<+\infty$*;*

(2) *if there exists such positive* ε *that*

$$a(t)-b(t)\geq\varepsilon,$$

then the Cauchy function $C(t,s)$ *of equation* (1) *satisfies the exponential estimate and the integral estimate*

$$\sup_{t\geq 0}\int_0^t C(t,s)\,ds\leq\frac{1}{\varepsilon}\,;$$

(3) *if there exists* $\lim_{t\to\infty}\{a(t)-b(t)\}=k$, *with* $k>0$, *then*

$$\lim_{t\to\infty}\int\limits_0^t C(t,s)\,ds=\frac{1}{k}\,.$$

Consider the equation

$$x''(t)+a(t)x(t-\tau)=f(t),\ \ t\in[0,+\infty),\ \ a(t)\ge a_*>0, \tag{3}$$

which is unstable. To stabilize its solution to the given "trajectory" $y(t)$ satisfying this equation, we choose the control in the form

$$u(t)=b(t)\big[x(t-\theta)-y(t-\theta)\big]. \tag{4}$$

Example 1. Stabilizing equation (3), where $a(t)\equiv a$, let us choose the control in the form (4) with constant coefficient $b(t)\equiv b$. We come to the study of the exponential stability of the equation

$$x''(t)+ax(t-\tau)-bx(t-\theta)=g(t),\ \ t\in[0,+\infty),$$

with constant coefficient and delays and $g(t)=f(t)+by(t-\theta)$. We can choose $\theta-\tau=\frac{1}{eb\theta}$ and

$$0<4\{a-b\}\le\frac{1}{e^2\theta^2}\,.$$

Example 2. The equation

$$x''(t)+a(t)x(t-\tau)=0,\ \ a(t)\to+\infty,\ \ t\in[0,+\infty),\ \ \tau=const,$$

where $a(t)\ge a_*>0$, possesses oscillating solutions with amplitudes tending to infinity that leads to the chaos in behavior of its solutions. This equation can also be stabilized by the control in form (4). Consider, for example, the equation

$$x''(t)+tx(t-\tau)=0,\ \ t\in[1,+\infty),\ \ \tau=const.$$

If we choose $b(t)=t-\Delta$, $\theta(t)=\tau+\frac{\gamma}{t}$, then the stabilization can be achieved by the control (4) with the parameters satisfying the inequalities

$$0<2\sqrt{\Delta}<\gamma<\frac{1}{\tau e}\,.$$

Example 3. Consider the equation

$$x''(t)+x(t)-bx(t-\theta)=0,\ \ t\in[0,+\infty),$$
$$x(\xi)=0,\ \ \xi<0.$$

It is clear from the definition of the Cauchy function that for $t<\theta$, this equation is equivalent to the ordinary differential equation

$$x''(t)+x(t)=0,\ \ t\in[0,\theta],$$

whose Cauchy function is $C(t,s)=\sin(t-s)$. Let us demonstrate that condition (2) is essential for positivity of the Cauchy function $C(t,s)$ in theorem, assuming that the numbers b and θ are chosen such that all other conditions of theorem are fulfilled. If $\pi<\theta$, then in the triangle $0\le s\le t<\theta$ its Cauchy function $C(t,s)=\sin(t-s)$ changes its sign.