The Control Problem of the Frequency Spectrum of Irregular Oscillations for Linear Systems

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For a long time, until the end of the 1940s, the investigation of periodic solutions of periodic differential systems was based on the conjecture of commensurability of periods of the solution and the system. Apparently, Massera was the first to indicate that this conjecture is wrong. In 1950, he showed that periodic differential system can have periodic solutions with irrational ratio of periods of the solution and the system [1]. Later, such periodic solutions were considered by J. Kurzweil and O. Vejvoda [2], N. P. Erugin [3], E. I. Grudo [4] and other authors. In what follows, such periodic solutions and the oscillations described by them are said to be strongly irregular [5].

Note that Mandelshtam and Papaleksi [6] studied the parametric influence on two-circuit parametric systems in the mid-1930s. In particular, for the case in which some capacity is included in the feed circuit of an electric motor to compensate for variable inductance, the following facts were justified: the rotation velocity of the electric motor is not synchronous with the supply current frequency; this velocity varies smoothly with the fundamental frequency of the oscillation contour. Unlike the ordinary parametric excitation, which takes place only for an integer frequency ratio, they obtained a new peculiar transformation of the motor frequency with practically arbitrary ratio to the circuit frequency. Thus, the possibility of excitation of oscillations at frequencies incommensurable with the frequency of the system parameter variation was shown.

Devices transforming the energy of a high-frequency oscillation source into low-frequency oscillations whose frequency is almost independent of the source frequency were developed at the beginning of the 1970s. For example, the paper [7] deals with the case in which a harmonic force with which the field in a capacitor acts on a flying charge has a frequency incommensurable with the frequency of fundamental oscillations of the charge. In this case, there can appear stable undamped oscillations at the natural frequency, i.e., strongly irregular oscillations. The conditions of a process in which the oscillations of a system are described by strongly irregular oscillations are referred to as an asynchronous mode [8]. Asynchronous modes in particular occur in linear differential systems. We state the problem of synthesis of asynchronous modes of linear systems as a control problem for the spectrum of irregular oscillations.

Consider the linear control system

$$\dot{x} = A(t)x + Bu, \quad t \in \mathbb{R}, \;\; x \in \mathbb{R}^n, \;\; n \geq 2, \tag{1}$$

where $A(t)$ is a continuous ω-periodic $n \times n$-matrix, B is a constant $n \times n$-matrix. We assume that the control is given in the form of a linear state feedback

$$u = U(t)x \tag{2}$$

with ω-periodic $n \times n$-matrix $U(t)$. The problem of finding the matrix $U(t)$ (the feedback coefficient) such that the closed system

$$\dot{x} = (A(t) + BU(t))x \tag{3}$$

has strongly irregular periodic solutions with a given frequency spectrum L (the objective set) will be called the problem of control of the frequency spectrum of irregular oscillations (asynchronous spectrum) with objective set L.

This problem is a version of the generalization of the spectrum assignment problem in the nonstationary case, but essentially differs from the problems of control [9, 10]. Note that, for regular oscillations, the choice of frequencies other than multiples of the frequencies of the right-hand side of system (1) is impossible.

Let $L = \{\lambda_1, \dots, \lambda'_r\}$ be an objective set of frequencies whose elements are pairwise distinct, commensurable with each other, and incommensurable with $2\pi/\omega$. Then there exists a maximum positive real number λ such that $\lambda_1, \dots, \lambda'_r$ are multiples of λ. Set $\Omega = 2\pi/\lambda$, then the ratio ω/Ω is irrational.

Consider the case of a singular matrix B, i.e. $\operatorname{rank} B = r < n$ $(n - r = d)$. One can assume that the first d rows of B are zero (otherwise it can always be achieved by a linear nonsingular time-independent transformation). Suppose that right upper $d \times r$ blocks of the averaged matrix $\widehat{A}$ are zero.

Let $\widetilde{A}^{(11)}_{d,d}$ and $\widetilde{A}^{(12)}_{d,r}$ be upper left and right blocks of the matrix $\widetilde{A}(t) = A(t) - \widehat{A}$ (the subscripts indicate the dimension). Assume that column bases of these blocks form a linearly independent set

$$\operatorname{rank}_{\mathrm{col}} \{\widetilde{A}^{(11)}_{d,d}, \widetilde{A}^{(12)}_{d,r}\} = r_1 + r_2 \quad \big(\operatorname{rank}_{\mathrm{col}} \widetilde{A}^{(11)}_{d,d} = r_1,\ \operatorname{rank}_{\mathrm{col}} \widetilde{A}^{(12)}_{d,r} = r_2\big).$$

By Q we denote a constant nonsingular $d \times d$ matrix such that the first $d_1 = d - r_1$ columns of the matrix $\widetilde{A}_{d,d}(t)Q$ are zero and the remaining columns are linearly independent. Let the left upper $d \times d$ block of the matrix $Q^{-1}\widehat{A}_{d,d}Q$ have p pairs of pure imaginary eigenvalues $\pm i\lambda_j$, $\lambda_j \in L$.

Then the following assertion holds.

Theorem. *Let the above assumptions are satisfied. The problem of control of the asynchronous spectrum with objective set L for system* (1) *with feedback* (2) *is solvable if and only if* $r_1 + r_2 < n$ *and* $|L| < p + [(r - r_2)/2]$.

References

[1] J. L. Massera, Remarks on the periodic solutions of differential equations. (Spanish) *Bol. Fac. Ingen. Montevideo* **4** (1950) (Ano 14), 37–45 *Publ. Inst. Mat. Estadistica* **2** (1950), 43–53.

[2] J. Kurzweil and O. Vejvoda, On the periodic and almost periodic solutions of a system of ordinary differential equations. (Russian) *Czechoslovak Math. J.* **5(80)** (1955), 362–370.

[3] N. P. Erugin, On periodic solutions of differential equations. (Russian) *Prikl. Mat. Meh.* **20** (1956), 148–152.

[4] È. I. Grudo and A. K. Demenchuk, On periodic solutions with incommensurable periods of linear inhomogeneous periodic differential systems. (Russian) *Differentsial'nye Uravneniya* **23** (1987), No. 3, 409–416.

[5] A. K. Demenchuk, Asynchronous Oscillations. Existence Conditions and Control. (Russian) *Lambert Academic Publishing, Saarbusken*, 2012.

[6] N. D. Papaleksi, On a Certain Case of Parametrically Related Systems. *J. Phys. Acad. Sc. USSR* **1** (1939), 373–379.

[7] D. I. Penner, Ya. B. Duboshinskiĭ, D. B. Duboshinskiĭ, and M. I. Kozakov, Oscillations with self-regulating interaction time. *Dokl. Akad. Nauk SSSR* **204** (1972), No. 5, 1065–1066.

[8] P. S. Landa and Ya. B. Duboshinskiĭ, Self-oscillatory systems with high-frequency energy sources. (Russian) *Uspekhi Fiz. Nauk* **158** (1989), No. 4, 729–742.

[9] P. Brunovský, Controllability and linear closed-loop controls in linear periodic systems. *J. Differential Equations* **6** (1969), 296–313.

[10] E. K. Makarov and S. N. Popova, Controllability of Asymptotic Invariants for Nonstationary Linear Systems. (Russian) *Minsk*, 2012.