Averaging Method in Optimal Control Problems for Systems of Ordinary Differential Equations

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The object of our investigation is an optimal control problem for the system of ordinary differential equations in standard Bogolyubov form:

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon X(t,x,u), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $\varepsilon > 0$ – small parameter, $x \in D$ – phase vector, D – domain in R^n, $u(t) \in U \subset R^m$ – control vector, $t \geq 0$, $T > 0$ – some constant, X – continuous of set of variables vector-function.

The idea of investigation consists in replacement of initial object (1) by simpler averaging object.

Our results are divided into two types:

(1) controls on asymptotically finite intervals (the order of $1/\varepsilon$);

(2) controls on semiaxis.

Let us denote by $x(t,u)$ a solution of system (1), corresponding to control $u(t)$.

We say that $u(t)$ are admissible controls, if

(1) $u(t) \in U$, where $t \geq 0$, $u(t)$ are measurable, locally Lebesgue integrable functions, where $t \geq 0$ and for every $u(t)$ there exists a constant $u_0 \in U$ such that $|u(t) - u_0| \leq \varphi(t)$, where $\varphi(t)$ doesn't depend on $u(t)$ and $\int\limits_0^\infty \varphi(t)dt < \infty$;

(2) a solution $x(t,u)$ of the Cauchy problem (1) is defined for every $t \in [0, \frac{T}{\varepsilon}]$.

The set of such controls is denoted by F.

The following problem is considered:

$$J_\varepsilon(u) = \Phi\Big(x\Big(\frac{T}{\varepsilon}, u\Big)\Big) \longrightarrow \inf, \tag{2}$$

where $\Phi(x)$ is some function.

Denote $J_\varepsilon = \inf\limits_{u(t) \in F} J_\varepsilon(u)$.

Let us consider the following optimal control problem, which is averaged to (1), (2):

$$\begin{aligned} \dot{y} &= \varepsilon X_0(y,u), \\ y(0) &= x_0, \end{aligned} \tag{3}$$

where

$$X_0(x,u) = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T X(t,x,u)\,dt \tag{4}$$

and

$$\overline{J}_\varepsilon(u) = \Phi(y(\frac{T}{\varepsilon},u)) \longrightarrow \inf\,. \tag{5}$$

Theorem 1. *Let us suppose that in the domain Q the following conditions are satisfied:*

(1) *$X(t,x,u)$ is measurable with respect to t and satisfies the Lipschitz condition with respect to x and u;*

(2) *solution $y = y(t,u_0)$, $y(0,u_0) = x_0$ is definite for every $t \geq 0$ and for every $u_0 \in U$ belongs to the domain D with some ρ-neighborhood;*

(3) *the limit (4) exists uniformly with respect to $x \in D$ and $u \in U$;*

(4) *function $\Phi(x)$ satisfies the Lipschitz condition with respect to $x \in D$;*

(5) *there exist optimal control of averaged problem $u^*(t,\varepsilon)$.*

Then for every $\eta > 0$ there exists $\varepsilon_0(\eta) > 0$ such that

(a) *for every $0 < \varepsilon < \varepsilon_0$, $J_\varepsilon > -\infty$;*

(b) *the following inequality is true*

$$\big|J_\varepsilon(u^*(t,\varepsilon)) - J_\varepsilon)\big| \leq \eta,$$

i.e. optimal control $u^*(t,\varepsilon)$ of averaged system is η-optimal control of precise system.

Consider the optimal control problem of the system of differential equations on semiaxis:

$$\begin{aligned} \dot{x} &= \varepsilon X(t,x,u), \\ x(0) &= x_0 \end{aligned} \tag{6}$$

with the control function

$$J(u) = \int\limits_0^\infty L(t,x,u)\,dt \longrightarrow \inf,$$

where $\varepsilon > 0$ – small parameter, $t \geq 0$, $x \in D$ – phase vector, D - domain in R^n, $u \in U \subset R^m$ – control vector, L satisfies the condition

$$\big|L(t,x,u) - L(t,y,u)\big| \leq \alpha(t)|x-y|, \ \text{ where } \int\limits_0^\infty \alpha(t)\,dt < \infty. \tag{7}$$

Let us consider that $u(t)$ are admissible controls, if

(a_1) $u(t)$ are measurable, locally integrable functions, where $t \geq 0$, $u(t) \in U$;

(b_1) for every $u(t) \in U$ there exists a constant $u_0 \in U$ such that $|u(t) - u_0| \leq \varphi(t)$, where $\varphi(t)$ doesn't depend on $u(t)$ and $\int\limits_0^\infty \varphi(t)dt < \infty$;

(c_1) there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ a solution of the Cauchy problem $x(t,u)$ is defined and unique for every $t \geq 0$;

(d_1) $|J(u)| < \infty$.

Let us consider the following optimal control problem which is averaged to (6):

$$\dot{y} = \varepsilon X_0(y, \overline{u}), \quad y(0) = x_0, \tag{8}$$

with the control functional

$$\overline{J}(\overline{u}) = \int_0^\infty L(t, y, \overline{u})\, dt,$$

where

$$X_0(x, u) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} X(s, x, u)\, dt, \quad t \geq 0. \tag{9}$$

The admissible controls for averaged system satisfy the same conditions (a_1)–(d_1). Let's suppose that the following conditions are performed for averaged system:

$(A1)$ solution $\overline{y}(\tau) = \overline{y}(\tau, u_0)$ of the averaged system

$$\frac{d\overline{y}}{d\tau} = X_0(\overline{y}, u_0), \quad \overline{y}(0) = x_0, \ \tau = \varepsilon t \tag{10}$$

is defined for every $\tau \geq 0$ and belongs to the domain D with some ρ - neighborhood, where ρ doesn't depend on u_0 for arbitrary constant control $u_0 \in U$;

$(A2)$ solution $\overline{y}(\tau)$ is equiasymptotically stable with respect to τ_0 and u_0.

The existence of η-optimal controls for precise system is proved in the following theorem with using of Lemma 3.

Theorem 2. *Let us suppose that in the domain* $Q = \{t \geq 0,\ x \in D \subset R^n,\ u \in U \subset R^m\}$ *the conditions of Lemma* 3 *are satisfied and there exists an optimal control* $\overline{u}^*$ *of averaged problem* (8)*. Then for every* $\eta > 0$ *there exists* $\varepsilon_0 = \varepsilon_0(\eta) > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$, $|J^*| < \infty$ *and the following inequality is true*

$$|J(\overline{u}^*) - J^*)| \leq \eta.$$