The Connection Between the Delayed Hopfield Network Model and the Wilson–Cowan Neural Field Model with Delay

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We consider the following delayed Hopfield network model (see e.g. [1])

$$\dot{z}_i(t,N) = -\alpha z_i(t,N) + \sum_{j=1}^{N} \omega_{ij}(N) f\big(z_j\big(t-\tau_{ij}(t,N),N\big)\big) + \mathrm{J}_i(t,N), \quad t>a, \quad i=1,\dots,N, \tag{1}$$

parameterized by a natural parameter N. Here at each natural N, $z_i(\,\cdot\,,N)$ are n-dimensional vector functions, $\omega_{ij}(N)$ are real $n\times n$-matrices (connectivities), $\tau_{ij}(\,\cdot\,,N)$ are nonnegative real-valued continuous functions (axonal delays), $f : R^n \to R^n$ are firing rate functions which are Lipschitz and bounded on R and $\mathrm{J}_i(\,\cdot\,,N)$ are continuous external input n-dimensional vector functions.

The initial conditions for (1) are given as

$$z_i(\xi,N) = \varphi_i(\xi,N), \quad \xi \le a, \quad i=1,\dots,N. \tag{2}$$

We use the general well-posedness result from the paper [2] to justify the convergence of a sequence of the delayed Hopfield equations (1) (with the initial conditions (2)) to the following Volterra integral equation involving spatio-temporal delay

$$\partial_t u(t,x) = -\alpha u(t,x) + \int\limits_{\Omega} \omega(x,y) f\big(u\big(s-\tau(t,x,y),y\big)\big)\, d_\nu(y) + J(t,x), \quad t>a, \quad x\in\Omega, \tag{3}$$

with the initial (prehistory) condition

$$u(\xi,x) = \varphi(\xi,x), \quad \xi\le a, \quad x\in\Omega. \tag{4}$$

The equation (3) generalizes the well-known neural field models introduced by Wilson and Cowan in [3, 4]. Here the function u represents the activity of a neural element at time t and position x. The connectivity kernel ω determines the coupling between elements at positions x and y. The non-negative activation function f gives the firing rate of a neuron with activity u. The non-negative function τ represents the time-dependent axonal delay effects in the neural field, which require a prehistory condition given by the function φ. The function $I(t,x)$ represents a variable external input.

The following assumptions will be imposed on the functions involved in (3) and (4):

(**A1**) The function $f : R^n \to R^n$ is continuous, bounded and Lipschitz.

(**A2**) The spatio-temporal delay $\tau : R\times\Omega\times\Omega \to [0,\infty)$ is a continuous function.

(**A3**) The initial (prehistory) function $\varphi : (-\infty,a]\times\Omega \to R^n$ is continuous.

(**A4**) For any $b>a$, the external input function $J : [a,b]\times\Omega \to R^n$ is uniform continuous.

(**A5**) The kernel function $\omega : \Omega\times\Omega \to R^n$ is continuous.

(**A6**) $\nu(\,\cdot\,)$ is the Lebesgue measure on Ω.

(**A7**) For any $b > a$,

$$\sup_{x\in\Omega} \int\limits_{\Omega} |\omega(x,y)|\, d_\nu(y) < \infty.$$

(**A8**) For any $b > a$,

$$\lim_{r\to\infty} \sup_{x\in\Omega} \int\limits_{\Omega-\Omega_r} |\omega(x,y)|\, d_\nu(y) = 0.$$

The following theorem shows the connection between the delayed Hopfield network model and the Wilson–Cowan neural field model with delay:

Theorem. *For each natural number N let $\{\Delta_i(N),\ i = 1,\dots,N\}$ be a finite family of open subsets of Ω satisfying the conditions*

$$\bigcup_{i=1}^{N} \overline{\Delta}_i(N) = \Omega_N \quad \text{and} \quad \lim_{N\to\infty} \mathit{mesh}\big\{\Delta_i(N),\ i = 1,\dots,N\big\} = 0.$$

Let $y_i(N)$ $(i = 1,\dots,N)$ be arbitrary points in $\Delta_i(N)$. Finally, let the assumptions (**A1**) – (**A8**) *be fulfilled.*

Then the sequence of the solutions $z_i(t,N)$ $(t \in R)$ of the initial value problem (1), (2), *where the coefficients are defined by*

$$\omega_{ij}(N) = \beta_i(N)\omega\big(y_i(N), y_j(N)\big), \quad \text{where } \beta_i(N) = \nu(\Delta_i(N)),$$
$$\tau_{ij}(t,N) = \tau(t, y_i(N), y_j(N)), \quad \mathrm{J}_i(t,N) = J(t, y_i(N)),$$

converges for any $b > a$ to the solution $u(t,x)$ $(t \in R,\ x \in \Omega)$ of the initial value problem (3), (4), *as $N \to \infty$, in the following sense:*

$$\lim_{N\to\infty} \Big\{ \sup_{t\in[a,b]} \Big\{ \sup_{1\le i\le N} \Big\{ \sup_{x\in\Delta_i(N)} \big|u(t,x) - z_i(t,N)\big| \Big\}\Big\}\Big\} = 0.$$

References

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