Asymptotic Behavior of Solutions of Essentially Nonlinear Differential Equations of the n-th Order

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The differential equation

$$y^{(n)} = \alpha_0 p(t) f\big(t, y, y', \dots, y^{(n-1)}\big) \prod_{i=0}^{n-1} \varphi_i(y^{(i)}), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\,^1 \to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i = 0, \dots, n)$ are continuous functions, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0, Y_i[\,^2$, or the interval $]Y_i, y_i^0]$, $f : [a, \omega[\, \times \Delta_{Y_0} \times \dots \times \Delta_{Y_{n-1}} \to]0, +\infty[$ is continuously differentiable function, that satisfies the conditions

$$\lim_{\substack{z_i \to Y_i \\ z_i \in \Delta_{Y_i}}} \frac{z_i \frac{\partial f}{\partial z_i}(t, z_1, \dots, z_{n-1})}{f(t, z_1, \dots, z_{n-1})} = 0 \ \ (i = \overline{1, n-1}),$$

$$\text{uniformly for } \ t \in [a, \omega[\,, \ \ z_j \in \Delta_{Y_j} \ \ (j \in \{1, \dots, n-1\} \setminus \{i\}),$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) \frac{\partial f}{\partial t}(t, z_1, \dots, z_{n-1})}{f(t, z_1, \dots, z_{n-1})} = 0 \ \text{ uniformly for } \ z_j \in \Delta_{Y_j} \ \ (j = \overline{1, n-1}),$$

is considered. We suppose also that every $\varphi_i(z)$ is regularly varying as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of index σ_i and $\sum\limits_{i=0}^{n-1} \sigma_i \neq 1$.

According to the type of the functions $\varphi_0, \dots, \varphi_{n-1}$ it is clear that the equation (1) is in some sense similar to the well-known differential equation

$$y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} |y^{(i)}|^{\sigma_i}. \tag{2}$$

We call the solution y of the equation (1) the $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solution, where $-\infty \le \lambda_{n-1}^0 \le +\infty$, if the next conditions take place

$$y^{(i)} : [t_0, \omega[\, \to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, \dots, n-1), \quad \lim_{t \uparrow \omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t)\, y^{(n-2)}(t)} = \lambda_{n-1}^0.$$

All $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solutions of the equation (2) were investigated in [2, 3]. Then for all $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solutions of the equation (1) the necessary and sufficient conditions of existence and asymptotic representations as $t \uparrow \omega$ were found in case $f(t, z_1, \dots, z_{n-1}) \equiv 1$ (see, for example, [4]). But it is clear that even slowly varying nonlinearities can not be represented as the product of functions of one variable. For equations of the type (1), that contain for example functions like $\exp(\sqrt{|\ln|tyy'||})$ or $\exp(\sqrt[m]{|\ln|t|y|^\mu y''||})$, the asymptotic representations of $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solutions were not before established. For the general case of equation (1) there are used some methods of investigations, that were perviously developed for the case

[1] If $\omega > 0$, we will take $a > 0$.
[2] If $Y_i = +\infty (Y_i = -\infty)$, we take $y_i^0 > 0$ $(y_i^0 < 0)$.

$f(t, z_1, \ldots, z_{n-1}) \equiv 1$. It is clear that equation (1) there may contain in the right part functions that are described before and many other slowly varying functions of many variables.

The cases $\lambda_0 \in \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}\}$ are singular by the studying of $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-solutions of the equation (1). $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-solutions, where $\lambda_0 \in \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}\}$ are regularly varying functions as $t \uparrow \omega$ of indexes $\{0, 1, \ldots, n-1\}$. To investigate such solutions we must put additional conditions on the functions $\varphi_0, \ldots, \varphi_{n-1}$ and the function p. The $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-solutions of the equation (1) in regular cases $\lambda_{n-1}^0 \in R \setminus \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}\}$ are established in this work. The $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-solutions of the equation (1) are regularly varying functions as $t \uparrow \omega$ of indexes different from $\{0, 1, \ldots, n-1\}$ if $\lambda_{n-1}^0 \in R \setminus \{0, 1, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}\}$. If $\lambda_0 = 1$, the $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-solutions of the equation (1) are rapidly varying functions as $t \uparrow \omega$.

Now we need the next subsidiary notations.

$$\gamma_0 = 1 - \sum_{j=0}^{n-1} \sigma_j, \quad \mu_n = \sum_{j=0}^{n-1} (n-j-1)\sigma_j, \quad \pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t - \omega & \text{if } \omega < +\infty, \end{cases}$$

$$\theta_i(z) = \varphi_i(z)|z|^{-\sigma_i}, \quad a_{0i} = (n-i)\lambda_{n-1}^0 - (n-i-1) \ \ (i = 1, \ldots, n),$$

$$C = \alpha_0 |\lambda_{n-1}^0 - 1|^{\mu_n} \prod_{k=0}^{n-2} \Big| \prod_{j=k+1}^{n-1} a_{0j} \Big|^{-\sigma_k} \operatorname{sign} y_{n-1}^0,$$

$$I_0(t) = \int_{A_\omega^0}^t C p(\tau) |\pi_\omega(\tau)|^{\mu_n} \, d\tau, \quad I_1(t) = \int_{A_\omega^1}^t \alpha_0 p(\tau) \, d\tau,$$

$$A_\omega^0 = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma_0} \, d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma_0} \, d\tau < +\infty, \end{cases} \qquad A_\omega^1 = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega p(\tau) \, d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega p(\tau) \, d\tau < +\infty, \end{cases}$$

$$J(t) = \int_{B_\omega}^t |\gamma_0 I_1(\tau)|^{\frac{1}{\gamma_0}} \, d\tau, \quad B_\omega = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}} \, d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}} \, d\tau < +\infty. \end{cases}$$

The following conclusions take place for equation (1).

Theorem 1. *The next conditions are necessary for the existence of* $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-*solutions* ($\lambda_{n-1}^0 \in \mathbb{R} \setminus \{0, 1, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}\}$) *of equation* (1)*:*

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) I_0'(t)}{I_0(t)} = \frac{\gamma_0}{\lambda_{n-1}^0 - 1}, \quad y_i^0 \lim_{t \uparrow \omega} |\pi_\omega(t)|^{\frac{a_{0i+1}}{\lambda_{n-1}^0 - 1}} = Y_i, \tag{3}$$

$$y_i^0 y_{i+1}^0 a_{0i+1} (\lambda_{n-1}^0 - 1) \pi_\omega(t) > 0 \ \ \text{as } \ t \in [a, \omega[\,, \tag{4}$$

where $y_n^0 = \alpha_0$, $i = \overline{0, n-1}$.

If the equation

$$\sum_{k=0}^{n-1} \sigma_k \prod_{i=k+1}^{n-1} a_{0i} \prod_{i=1}^{k} (a_{0i} + \lambda) = (1 + \lambda) \prod_{i=1}^{n-1} (a_{0i} + \lambda)$$

has no roots with zero real part, then conditions (3) *and* (4) *are sufficient for the existence of* $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$-*solutions of equation* (1)*. For any such solution the next asymptotic rep-*

resentations as $t \uparrow \omega$

$$\frac{|y^{(n-1)}(t)|^{\gamma_0}}{f(t,y,y',\dots,y^{(n-1)})\prod\limits_{j=0}^{n-1}\theta_j(y^{(j)}(t))} = \gamma_0 I_0(t)[1+o(1)],$$

$$\frac{y^{(i)}(t)}{y^{(n-1)}(t)} = \frac{[(\lambda_{n-1}^0-1)\pi_\omega(t)]^{n-i-1}}{\prod\limits_{j=i+1}^{n-1} a_{0j}}[1+o(1)],$$

where $i=\overline{0,n-2}$*, take place.*

Theorem 2. *The next conditions are necessary for the existence of* $P_\omega(Y_0,Y_1,\dots,Y_{n-1},1)$*-solutions of equation* (1)*:*

$$\lim_{t\uparrow\omega}\frac{I_1'(t)J(t)}{I_1(t)J'(t)} = \gamma_0, \quad y_i^0\lim_{t\uparrow\omega}|I_1(t)|^{\frac{1}{\gamma_0}} = Y_i, \tag{5}$$

$$\alpha_0 y_{n-2}^0 > 0, \quad y_i^0 y_{i+1}^0 J(t) > 0 \ \ as \ \ t\in[a,\omega[\,, \tag{6}$$

where $i=\overline{0,n-1}$.

If the equation $\sum\limits_{k=0}^{n-1}\sigma_k(1+\lambda)^k = (1+\lambda)^n$ *has no roots with zero real part, then conditions* (5) *and* (6) *are sufficient for the existence of* $P_\omega(Y_0,Y_1,\dots,Y_{n-1},1)$*-solutions of equation* (1)*. For any such solution the asymptotic representations as* $t\uparrow\omega$

$$\frac{|y^{(n-1)}(t)|^{\gamma_0}}{f(t,y,y',\dots,y^{(n-1)})\prod\limits_{j=0}^{n-1}\theta_j(y^{(j)}(t))} = \gamma_0 I_1(t)\Big|\frac{J(t)}{J'(t)}\Big|^{\mu_n}\operatorname{sign} y_{n-1}^0[1+o(1)],$$

$$\frac{y^{(i)}(t)}{y^{(n-1)}(t)} = \Big(\frac{J(t)}{J'(t)}\Big)^{n-i-1}[1+o(1)],$$

where $i=\overline{0,n-2}$*, take place.*

Let us notice that if $\sum\limits_{i=0}^{n-2}|\sigma_i| < |1-\sigma_{n-1}|$, the conditions (3) and (4) are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},\lambda_0)$-solutions ($\lambda_{n-1}^0\in\mathbb{R}\setminus\{0,1,\frac12,\frac23,\dots,\frac{n-2}{n-1}\}$) of equation (1) and the conditions (5) and (6) are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},1)$-solutions of (1).

References

[1] E. Seneta, Regularly varying functions. *Lecture Notes in Mathematics*, Vol. 508. *Springer-Verlag, Berlin–New York*, 1976.

[2] V. M. Evtukhov, A class of monotone solutions of an nth-order nonlinear differential equation of Emden–Fowler type. (Russian) *Soobshch. Akad. Nauk Gruzii* **145** (1992), No. 2, 269–273.

[3] V. M. Evtukhov, Asymptotic representations of monotone solutions of an nth-order nonlinear differential equation of Emden–Fowler type. (Russian) *Dokl. Akad. Nauk* **324** (1992), No. 2, 258–260; translation in *Russian Acad. Sci. Dokl. Math.* **45** (1992), No. 3, 560–562 (1993).

[4] M. A. Bilozerowa and V. M. Evtukhov, Asymptotic representations of solutions of the differential equation $y^{(n)} = \alpha_0 p(t)\prod\limits_{i=0}^{n-1}\varphi_i(y^{(i)})$. *Miskolc Math. Notes* **13** (2012), No. 2, 249–270.