Difference Schemes for One Fully Nonlocal Boundary-Value Problem

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We study fully nonlocal problem for the Poisson equation when the classical boundary condition is not given on any part of the boundary of the integration domain. The paper represents generalization of investigation carried out by the authors in [1, 2].

We investigate finite difference scheme for the equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}=-f(x),\ \ x\in\Omega=(0,l_1)\times(0,l_2), \tag{1}$$

together with the integral constraints

$$\int\limits_0^{\xi} u(x)\,dx_\alpha=0,\quad \int\limits_{l_\beta-\xi}^{l_\beta} u(x)\,dx_\alpha=0,\quad 0\le x_\beta\le l_\beta,\ \ \xi\le l_\alpha/2,\ \ \beta=3-\alpha,\ \ \alpha=1,2. \tag{2}$$

We assume that the solution u of the nonlocal boundary-value problem $(1),(2)$ belongs to the Sobolev space $W_2^m(\Omega)$, $m>1$.

Consider the grid domains: $\overline{\omega}_\alpha=\{x_{\alpha,i_\alpha}=i_\alpha h;\ i_\alpha=0,1,\dots,n_\alpha,\ h=l_\alpha/n_\alpha\}$, $\alpha=1,2$, $\overline{\omega}=\overline{\omega}_1\times\overline{\omega}_2$. Let $\xi=(m+\theta)h$, $0\le\theta<1$.

Let H be the set of all discrete functions $v=v(x)$, defined on the grid $\overline{\omega}$ and satisfying conditions

$$\check{\mathcal{P}}_j^{(1)}(v)=0,\ \ \hat{\mathcal{P}}_j^{(1)}(v)=0,\ \ 0\le j\le n_2,\quad \check{\mathcal{P}}_i^{(2)}(v)=0,\ \ \hat{\mathcal{P}}_i^{(2)}(v)=0,\ \ 1\le i\le n_1-1,$$

where

$$\begin{aligned}
\check{\mathcal{P}}_j^{(1)}(v)&:=\sum_{k=0}^{m}hv_{kj}-\frac{h}{2}\,(y_{0j}+v_{mj})+\frac{\theta h}{2}\big((2-\theta)v_{mj}+\theta v_{m+1,j}\big),\\
\check{\mathcal{P}}_i^{(2)}(v)&:=\sum_{k=0}^{m}hv_{ik}-\frac{h}{2}\,(v_{i0}+v_{im})+\frac{\theta h}{2}\big((2-\theta)v_{im}+\theta v_{i,m+1}\big),\\
\hat{\mathcal{P}}_j^{(1)}(v)&:=\sum_{k=n_1-m}^{n_1}hv_{kj}-\frac{h}{2}\,(v_{n_1-m,j}+v_{n_1j})+\frac{\theta h}{2}\big((2-\theta)v_{n_1-m,j}+\theta v_{n_1-m-1,j}\big),\\
\hat{\mathcal{P}}_i^{2)}(v)&:=\sum_{k=n_2-m}^{n_2}hv_{ik}-\frac{h}{2}\,(v_{i,n_2-m}+v_{in_2})+\frac{\theta h}{2}\big((2-\theta)v_{i,n_2-m}+\theta v_{i,n_2-m-1}\big).
\end{aligned}$$

We approximate the problem $(1),(2)$ by the difference scheme

$$U_{\overline{x}_1x_1}+U_{\overline{x}_2x_2}=-\varphi(x),\ \ x\in\omega,\ \ U\in H,\ \ \varphi=T_1T_2f, \tag{3}$$

where U_{x_α}, $U_{\overline{x}_\alpha}$ denote forward and backward difference quotients in x_α directions respectively, and T_1, T_2 – some averaging operators.

An a priori estimate of the solution of the difference scheme (3) is obtained with the help of energy inequality method, from which it follows the unique solvability of the scheme.

To estimate the truncation error , we apply the well-known technique [3], which uses the generalized Bramble–Hilbert lemma.

It is proved that the discretization error of the difference scheme (3) in the discrete weighted W_2^1-norm is determined by the estimate

$$\|U-u\|_{W_2^1(\omega,\rho)} \le ch^{s-1}\|u\|_{W_2^s(\Omega)}, \quad 1 < s \le 3.$$

References

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