A Discussion of Some Sharp Constants in Oscillation and Stability Theory of Delay Differential Equations

Leonid Berezansky

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, Israel

Elena Braverman

Department of Mathematics and Statistics, University of Calgary, Calgary, Canada
E-mail: maelena@math.ucalgary.ca

Başak Karpuz

Department of Mathematics, Faculty of Science and Arts, ANS Campus, Afyon Kocatepe University, Afyonkarahisar, Turkey

For the equation with variable delays $x'(t) = -\sum\limits_{j=1}^{m} a_j x(t - h_j(t))$, where $a_j > 0$, $0 \le h_j(t) \le q_j$, the inequalities $\sum\limits_{j=1}^{m} a_j q_j \le 3/2$ and $\sum\limits_{j=1}^{m} a_j q_j < 3/2$ are sufficient for uniform and exponential stability, respectively [1]. If all $h_j(t)$ are constant, then $3/2$ can be replaced by $\pi/2$ [2].

However, these results are not valid if the coefficients are not constant. For the equation $x'(t) = -\sum\limits_{j=1}^{m} a_j(t) x(t - h_j(t))$, where $0 \le a_j(t) < \alpha_j$, $0 \le h_j(t) \le q_j$, the inequality $\sum\limits_{j=1}^{m} \alpha_j q_j \le 1$ is sufficient for uniform stability, and constant 1 is sharp, as proved in the paper [1].

In this talk, we answer the following question: does there exist a number $A > 0$ such that the inequality

$$\limsup_{t\to\infty} \int\limits_{t-h(t)}^{t} a(s)\,ds \ge A \tag{1}$$

implies instability of equation

$$x'(t) = -a(t)x(t - h(t)) \tag{2}$$

with one variable delay and a positive coefficient?

Another object of the talk is to discuss constants which lead to either oscillation or non-oscillation. For delay differential equations the following result is well known [3]:

If

$$\limsup_{t\to\infty} \int\limits_{t-\max_k h_k(t)}^{t} \sum_{j=1}^{m} a_j^{+}(s)\,ds < \frac{1}{e}\,, \tag{3}$$

then there exists an eventually positive solution of the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x(t - h_k(t)) = 0. \tag{4}$$

Here $1/e$ is the best possible constant since the equation $\dot{x}(t) + x(t - \tau) = 0$ is oscillatory for $\tau > 1/e$.

In the monograph [3] for the equation

$$\dot{x}(t) + a(t)x(t - \tau) = 0, \;\; a(t) \ge 0, \;\; \tau > 0, \tag{5}$$

the authors constructed a counterexample which shows that condition (3) is not necessary for non-oscillation of equation (4).

By [3, Theorem 3.4.3], the inequality

$$\limsup_{t\to\infty} \int\limits_{t-\min_k h_k(t)}^{t} \sum_{j=1}^{m} a_j(s)\, ds < 1 \tag{6}$$

is necessary for non-oscillation of equation (4), where $a_j(t) \geq 0$, however the result is valid for $h(t)$ monotone only (in particular, $h(t) = t - \tau$).

Let us start with oscillation. First, we present sufficient non-oscillation conditions for equation (4) when the number $\limsup\limits_{t\to\infty} \int\limits_{t-\min_k h_k(t)}^{t} \sum\limits_{j=1}^{m} a_j(s)\, ds$ is between $1/e$ and 1.

Consider equation (4) with constant delays

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x(t-\tau_k) = 0, \quad \tau_k > 0. \tag{7}$$

Theorem 1 ([4]). *Suppose that there exists* $n_0 \geq 0$ *and a sequence* $\{\lambda_j\}_{j=n_0-1}^{\infty}$ *of positive numbers such that* $\sum\limits_{k=1}^{m} a_k^+(t) \leq \lambda_n e^{-[\lambda_{n-1}(n\tau - t) + \lambda_n(t-(n-1)\tau)]}$, $(n-1)\tau < t \leq n\tau$, $n \geq n_0$, *where* $\tau = \max_k \tau_k$. *Then equation* (7) *is non-oscillatory.*

Remark 1. By applying comparison theorems [3], Theorem 1 can be extended to equations with variable delays $0 \leq h_k(t) \leq \tau_k$.

Theorem 2 ([4]). *For any* $\alpha \in (1/e, 1)$ *there exists a non-oscillatory equation* (5) *with* $a(t) \geq 0$ *such that* $\limsup\limits_{t\to\infty} \int\limits_{t-\tau}^{t} a(s)\, ds = \alpha$.

Second, we give a negative answer to the following question: does there exist constants A and B such that

$$\limsup_{t\to\infty} \int\limits_{h(t)}^{t} a(u)\, du > A \tag{8}$$

implies oscillation of equation (2).

Theorem 3 ([5]). *There is no constant* $A > 0$ *such that* (8) *implies oscillation of* (2) *for arbitrary* $h(t) \leq t$.

Third, we present an explicit oscillation test in the terms of the maximal value of the deviated argument

$$g(t) := \sup_{s\leq t} h(s), \quad t \geq 0. \tag{9}$$

Clearly, $g(t)$ is nondecreasing, and $h(t) \leq g(t)$ for all $t \geq 0$.

Theorem 4 ([5]). *Assume that* $\sup\left\{t \geq 0 : \int\limits_{g(t)}^{t} a(u) \exp\left\{\int\limits_{h(u)}^{g(t)} a(v)\, dv\right\} du \geq 1\right\} = \infty$ *for* $g(t)$ *defined by* (9). *Then every solution of* (2) *is oscillatory.*

Theorem 4 immediately implies the following result.

Theorem 5 ([5])**.** *If* $\limsup_{t\to\infty} \int_{g(t)}^{t} a(u) \exp\Big\{ \int_{h(u)}^{g(t)} a(v)\,dv \Big\}\,\mathrm{d}u > 1$*, then every solution of* (2) *is oscillatory.*

The following result allows to expand the set of constants in (6) such that equation (2) may be stable, up to the limit of 2.

Lemma 1 ([6])**.** *Suppose* $a(t) \ge 0$*,* $\liminf_{t\to\infty} a(t) > 0$*,* $\limsup_{t\to\infty} h(t) < \infty$*, and there exists* $r(t) \ge 0$ *such that the equation*

$$\dot{x}(t) + a(t)x(t - r(t)) = 0 \tag{10}$$

is non-oscillatory. If $\limsup_{t\to\infty} \Big| \int_{t-h(t)}^{t-r(t)} a(s)\,ds \Big| < 1$*, then equation* (2) *is exponentially stable.*

By choosing an appropriate $r(t) = \tau$ and the same coefficient as in the proof of Theorem 2, we can for each $\alpha < 1$ construct non-oscillatory equation (10) such that

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} a(s)\,ds = \alpha.$$

Further, applying Lemma 1, for any $\beta < 2$ we can construct an exponentially stable equation with $\limsup_{t\to\infty} \int_{t-\tau}^{t} a(s)\,ds = \beta$, see [4].

However, this is still an open problem whether there exists $A > 0$ such that the inequality $\liminf_{t\to\infty} \int_{t-h(t)}^{t} a(s)\,ds > A$ implies instability of equation (2) with $a(t) \ge 0$ or not.

References

[1] T. Krisztin, On stability properties for one-dimensional functional-differential equations. *Funkcial. Ekvac.* **34** (1991), No. 2, 241–256.

[2] T. Krisztin, Stability for functional-differential equations and some variational problems. *Tohoku Math. J. (2)* **42** (1990), No. 3, 407–417.

[3] I. Győri and G. Ladas, Oscillation theory of delay differential equations. With applications. *Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York*, 1991.

[4] L. Berezansky and E. Braverman, On some constants for oscillation and stability of delay equations. *Proc. Amer. Math. Soc.* **139** (2011), No. 11, 4017–4026.

[5] E. Braverman and B. Karpuz, On oscillation of differential and difference equations with non-monotone delays. *Appl. Math. Comput.* **218** (2011), No. 7, 3880–3887.

[6] L. Berezansky and E. Braverman, Explicit exponential stability conditions for linear differential equations with several delays. *J. Math. Anal. Appl.* **332** (2007), No. 1, 246–264.