Sufficiency Conditions for the Asymptotic Stability of Solutions of Linear Homogeneous Nonautonomous Differential Equation of Second-Order

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In this paper we consider the problem on the stability of the real linear homogeneous differential equation (LHDE) of second order

$$y'' + p(t)y' + q(t)y = 0 \ \ (t \in I), \tag{1}$$

provided that the roots $\lambda_i(t)$ $(i = \overline{1,2})$ of the characteristic equation

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

are such that

$$\lambda_i(t) < 0 \ \ (t \in I), \quad \int\limits_{t_0}^{+\infty} \lambda_i(t)\, dt = -\infty \ \ (i = \overline{1,2}), \tag{2}$$

and there are finite or infinite limits $\lim\limits_{t\to+\infty} \lambda_i(t)$ $(i = \overline{1,2})$. Stability of the equation (1) is investigated by the way of its reduction to the equivalent system, which is led to an almost triangular form with the help of the linear transformation. Below we give the obtained by us results.

Theorem 1. *In the case of* $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) = o(1)$ *the trivial solution of the equation* (1) *is asymptotically stable. It is sufficient to assume that* $p(t), q(t) \in C_I$.

This theorem follows from the results of Lyapunov A. M.

Theorem 2. *Let the condition* (2) *hold for* $i = 2$, *and let:*

(1) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) = o(1)$;

(2) $\dfrac{\lambda_1'(t)}{\lambda_2(t)} = o(1)$.

Then the trivial solution of the equation (1) *is asymptotically stable.*

Theorem 3. *Let the condition* (2) *hold, and let:*

(1) $\lambda_i(t) = o(1)$ $(i = 1, 2)$;

(2) $\dfrac{\lambda_1'(t)}{\lambda_1^2(t)} = o(1)$ (*or* $\dfrac{\lambda_2'(t)}{\lambda_2^2(t)} = o(1)$), $\dfrac{\lambda_1(t)}{\lambda_2(t)} = O(1)$.

Then the trivial solution of the equation (1) *is asymptotically stable.*

Theorem 4. *Let the following conditions be fulfilled:*

(1) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) \to -\infty$, $\lambda_2(t) < 0$ *at* I;

(2) $\lambda_1'(t)$ *is bounded at* $t \to +\infty$.

Then the trivial solution of the equation (1) *is asymptotically stable.*

Theorem 5. *Let the condition* (2) *hold for* $i = 1$, *and let:*

(1) $\lambda_1(t) = o(1)$, $\lambda_2(+\infty) = -\infty$;

(2) $\dfrac{\lambda_1'(t)}{\lambda_1^2(t)} = o(1)$.

Then the trivial solution of the equation (1) *is asymptotically stable.*

Theorem 6. *Let the following conditions hold:*

(1) $\lambda_i(+\infty) = -\infty$ $(i = 1, 2)$;

(2) $\dfrac{\lambda_1'(t)}{\lambda_1^2(t)} = o(1)$ *or* $\dfrac{\lambda_2'(t)}{\lambda_2^2(t)} = o(1))$, $\dfrac{\lambda_1(t)}{\lambda_2(t)} = O(1)$.

Then the trivial solution of the equation (1) *is asymptotically stable.*

In all the above cases we have also obtained estimates for the solutions of the equation (1).

References

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