On Existence of Quasi-Periodic Solutions to a Nonlinear Higher-Order Differential Equation

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1 Introduction

The paper is devoted to the existence of oscillatory and non-oscillatory quasi-periodic, in some sense, solutions to the higher-order Emden–Fowler type differential equation

$$y^{(n)} + p_0 \, |y|^k \operatorname{sgn} y = 0, \quad n > 2, \;\; k \in \mathbb{R}, \;\; k > 1, \;\; p_0 \neq 0. \tag{1}$$

A lot of results about the asymptotic behavior of solutions to (1) are described in detail in [1]. Results about the existence of solutions with special asymptotic behavior are contained in [2]–[8].

2 On Existence of Quasi-Periodic Oscillatory Solutions

Put

$$\alpha = \frac{n}{k-1}\,. \tag{2}$$

Theorem 1. *For any integer $n > 2$ and real $k > 1$ there exists a non-constant periodic function $h(s)$ such that for any $p_0 > 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = p_0^{\frac{1}{k-1}} (x^* - x)^{-\alpha} h(\log(x^* - x)), \;\; -\infty < x < x^* \tag{3}$$

is a solution to equation (1)*.*

Corollary 1. *For any integer even $n > 2$ and real $k > 1$ there exists a non-constant periodic function $h(s)$ such that for any $p_0 > 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = p_0^{\frac{1}{k-1}} (x - x^*)^{-\alpha} h(\log(x - x^*)), \;\; x^* < x < \infty \tag{4}$$

is a solution to equation (1)*.*

Corollary 2. *For any integer odd $n > 2$ and real $k > 1$ there exists a non-constant periodic function $h(s)$ such that for any $p_0 < 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = |p_0|^{\frac{1}{k-1}} (x - x^*)^{-\alpha} h(\log(x - x^*)), \;\; x^* < x < \infty \tag{5}$$

is a solution to equation (1)*.*

3 On Existence of Positive Solutions with Non-power Asymptotic Behavior

The existence of such non-oscillatory solutions was also proved.

For equation (1) with $p_0 = -1$ it was proved [4] that for any N and $K > 1$ there exist an integer $n > N$ and $k \in \mathbf{R}$ such that $1 < k < K$ and equation (1) has a solution of the form

$$y = (x^* - x)^{-\alpha} h(\log(x^* - x)), \tag{6}$$

where α is defined by (2) and h is a positive periodic non-constant function on $\mathbf{R}$.

A similar result was also proved [4] about Kneser solutions, i.e. those satisfying $y(x) \to 0$ as $x \to \infty$ and $(-1)^j y^{(j)}(x) > 0$ for $0 \le j < n$. Namely, if $p_0 = (-1)^{n-1}$, then for any N and $K > 1$ there exist an integer $n > N$ and $k \in \mathbf{R}$ such that $1 < k < K$ and equation (1) has a solution of the form

$$y(x) = (x - x_*)^{-\alpha}\ h(\log(x - x_*)),$$

where h is a positive periodic non-constant function on $\mathbf{R}$.

Still it was not clear how large n should be for the existence of that type of positive solutions.

Theorem 2 ([8])**.** *If* $12 \le n \le 14$, *then there exists* $k > 1$ *such that equation* (1) *with* $p_0 = -1$ *has a solution* $y(x)$ *such that*

$$y^{(j)}(x) = (x^* - x)^{-\alpha-j} h_j(\log(x^* - x)), \quad j = 0, 1, \dots, n-1,$$

where α *is defined by* (2) *and* h_j *are periodic positive non-constant functions on* $\mathbf{R}$.

Remark. Computer calculations give approximate values of α. They are, with the corresponding values of k, as follows:

if $n = 12$, then $\alpha \approx 0.56$, $k \approx 22.4$;

if $n = 13$, then $\alpha \approx 1.44$, $k \approx 10.0$;

if $n = 14$, then $\alpha \approx 2.37$, $k \approx 6.9$.

Corollary 3 ([8])**.** *If* $12 \le n \le 14$, *then there exists* $k > 1$ *such that equation* (1) *with* $p_0 = (-1)^{n-1}$ *has a Kneser solution* $y(x)$ *satisfying*

$$y^{(j)}(x) = (x - x_0)^{-\alpha-j} h_j(\log(x - x_0)), \quad j = 0, 1, \dots, n-1,$$

with periodic positive non-constant functions h_j *on* $\mathbf{R}$.

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