On a Two-Point Boundary Value Problem for Systems of Nonlinear Generalized Ordinary Differential Equations with Singularities

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For the nonlinear generalized system with singularities

$$dx_i = f_i(t, x_1, \dots, x_n)da_i(t) \ \text{ for } \ t \in [a,b] \ \ (i = 1, \dots, n), \tag{1}$$

we consider the two-point boundary value problem

$$x_i(a+) = 0 \ \ (i = 1, \dots, n_0), \ \ x_i(b-) = 0 \ \ (i = n_0 + 1, \dots, n), \tag{2}$$

where $-\infty < a < b < +\infty$, $n_0 \in \{1, \dots, n\}$, $x_1, \dots, x_n$ are the components of the desired solution x; $a_i : [a,b] \to \mathbb{R}$ is a nondecreasing function, and $f_i :]a,b[\times \mathbb{R}^n \to \mathbb{R}$ is a function belonging to the local Carathéodory class $\mathrm{Car}_{loc}(]a,b[\times \mathbb{R}^n, \mathbb{R}; a_i)$ for every $i \in \{1, \dots, n\}$.

We investigate the question of solvability of the problem (1), (2), when the system (1) has singularities. Singularity is understood in a sense that the components of the vector-function f may have non-integrable components on the boundary points a and b, in general.

The interest to the theory of generalized ordinary differential equations has been motivated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see e.g. [1, 2] and references therein).

Basic notation and definitions. $\mathbb{N}_{1n_0} = \{1, \dots, n_0\}$, $\mathbb{N}_{2n_0} = \{1 + n_0, \dots, n\}$. $\mathbb{R} =]-\infty, +\infty[$. $\mathbb{R}^{n\times m}$ is the set of all real $n \times m$-matrices. $\mathbb{R}_+^{n\times m}$ is the set of all real nonnegative $n \times m$-matrices. $\mathbb{R}^n = R^{n\times 1}$. $r(X)$ the spectral radius of $X \in \mathbb{R}^{n\times n}$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function X at the point t. $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a,b], \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ with bounded variation components. $\mathrm{BV}_{loc}(]a,b[\,, \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X :]a,b[\to \mathbb{R}^{n\times m}$ with bounded variation components on every close interval from $[a,b]$.

If $\alpha \in \mathrm{BV}([a,b], \mathbb{R})$ has no more than a finite number of points of discontinuity, and $m \in \{1, 2\}$, then $D_{\alpha m} = \{t_{\alpha m1}, \dots, t_{\alpha m n_{\alpha m}}\}$ $(t_{\alpha m1} < \dots < t_{\alpha m n_{\alpha m}})$ is the set of all points from $[a,b]$ for which $d_m\alpha(t) \neq 0$, and $\mu_{\alpha m} = \max\{d_m\alpha(t) : \ t \in D_{\alpha m}\}$ $(m = 1, 2)$. If $\beta \in \mathrm{BV}([a,b], \mathbb{R})$, then

$$\nu_{\alpha m\beta j} = \max\bigg\{ d_j\beta(t_{\alpha ml}) + \sum_{t_{\alpha m\, l+1-m} < \tau < t_{\alpha m\, l+2-m}} d_j\beta(\tau) : \ l = 1, \dots, n_{\alpha m} \bigg\}$$

$(j, m = 1, 2)$; here $t_{\alpha 20} = a - 1$, $t_{\alpha 1 n_{\alpha 1}+1} = b + 1$.

g_c is the continuous part of a function $g : [a,b] \to \mathbb{R}$, and $D_g = \{t \in [a,b] : d_1g(t) + d_2g(t) \neq 0\}$.

Integrals are understood in the Kurzweil–Stieltjes sense (see, [2]).

$L^p_{loc}([a,b], \mathbb{R}; g)$ $(1 \le p < +\infty)$ and $L^{+\infty}_{loc}([a,b], \mathbb{R}; g)$ are the standard spaces of functions.

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}$, then $\mathrm{Car}([a,b] \times D_1, D_2; g)$ is the Carathéodory class, i.e., the set of all mappings $f : [a,b] \times D_1 \to D_2$ such that:(i) the function $f(\,\cdot\,, x) : [a,b] \to D_2$ is $\mu(g)$-measurable for every $x \in D_1$; (ii) the function $f(t, \cdot\,) : D_1 \to D_2$ is continuous for $\mu(g)$-almost every $t \in [a,b]$, and $\sup\big\{|f(\,\cdot\,, x)| : \ x \in D_0\big\} \in L([a,b], R; g)$ for every compact $D_0 \subset D_1$; $\mathrm{Car}_{loc}(]a,b[\times D_1, D_2; g)$ is the set of all mappings $f :]a,b[\times D_1 \to D_2$ the restriction of which on every closed interval $[c,d]$ of $]a,b[$ belongs to $\mathrm{Car}([c,d] \times D_1, D_2; g)$.

We assume that $a_i : I_i \to \mathbb{R}$ $(i = 1, \dots, n)$ are nondecreasing functions, and $f_i \in \mathrm{Car}_{loc}(I_i \times \mathbb{R}^n, \mathbb{R}^n; a_i)$ $(i = 1, \dots, n)$, where $I_i =]a, b]$ if $i \in \mathbb{N}_{1n_0}$ and $I_i = [a, b[$ if $i \in \mathbb{N}_{2n_0}$.

A vector-function $(x_i)_{i=1}^n$, $x_i \in \mathrm{BV}_{loc}(I_i, \mathbb{R})$ $(i = 1, \dots, n)$, is said to be a solution of the system (1) if $x_i(t) = x_i(s) + \int\limits_s^t f_l(\tau, x_1(\tau), \dots, x_n(\tau))\, da_i(\tau)$ for $s < t$; $s, t \in I_i$ $(i = 1, \dots, n)$.

A solution $(x_i)_{i=1}^n$ of the system (1) is said to be a solution of the problem (1), (2) if one-sided limits $x_i(a+)$ $(i \in \mathbb{N}_{1n_0})$ and $x_i(b-)$ $(i \in \mathbb{N}_{2n_0})$ exist and the equalities (2) hold.

Let $b_{il} : I_i \to \mathbb{R}$ $(i, l = 1, \dots, n)$ be nondecreasing functions. A vector-function $(x_i)_{i=1}^n$, $x_i \in \mathrm{BV}(I_i, \mathbb{R})$ $(i = 1, \dots, n)$, is said to be a solution of the system of differential inequalities $dx_i(t) \le \sum\limits_{l=1}^n x_l(t)\, db_{il}(t)$ for $t \in I_i$ $(i = 1, \dots, n)$, if it satisfies the corresponding integral inequalities.

We assume that $\det(1 + (-1)^j d_j a_i(t)) \neq 0$ for $t \in I_i$ $(j = 1, 2;\ i = 1, \dots, n)$. This inequalities guarantee the unique solvability of the Cauchy problem for the corresponding equation. By $\gamma_\beta(\,\cdot\,, s)$ we denote the unique solution of the Cauchy problem $d\gamma(t) = \gamma(t) d\beta(t)$, $\gamma(s) = 1$.

Definition. A matrix-function $C = (c_{il})_{i,l=1}^n$, $c_{il} \in \mathrm{BV}_{loc}(I_i, \mathbb{R}_+)$ $(i, l = 1, \dots, n)$ belongs to the set $\mathcal{U}(a+, b-; a_1, \dots, a_n; n_0)$ if the system

$$\operatorname{sgn}\Big(n_0 + \frac{1}{2} - i\Big)\, dx_i(t) \le \sum_{l=1}^n c_{il}(t) x_l(t)\, da_i(t) \ \text{ for } \ t \in I_i \ (i = 1, \dots, n)$$

has no nontrivial nonnegative solution satisfying the condition (2).

Theorem. *Let*

$$f_i(t, x_1, \dots, x_n) \operatorname{sgn}\Big(\Big(n_0 + \frac{1}{2} - i\Big) x_i\Big) \le -b_i(t)|x_i| +$$
$$+ \sum_{l=1}^n \eta_{il}(t)|x_l| \ \textit{ for } \ \mu(a_{ic})\text{-a.e. } t \in I_i \ \textit{ and for all } \ t \in D_{a_i},\ \ x_k \in \mathbb{R} \ \ (i, k = 1, \dots, n), \tag{3}$$

$$(-1)^j f_i(t, x_1, \dots, x_n) d_2 a_i(t) \operatorname{sgn}\Big(x_i + (-1)^j f_i(t, x_1, \dots, x_n) d_j a_i(t)\Big) \le$$
$$\le -b_i(t)|x_i| + \sum_{l=1}^n \eta_{il}(t)|x_l| \ \textit{ for } \ t \in I_i \ \ (j = 1, 2;\ \ i \in \mathbb{N}_{3-j\, n_0}), \tag{4}$$

where $\eta_{il} \in L_{loc}(I_i, \mathbb{R}; a_i)$, $b_i \in L_{loc}(I_i, \mathbb{R}_+; a_i)$ $(i, l = 1, \dots, n)$. *Let, moreover,* $C = (c_{il})_{i,l=1}^n \in \mathcal{U}(a+, b-; a_1, \dots, a_n; n_0)$,

$$\lim_{t \to a+} b_i(t) d_2 a_i(t) < 1, \ \ \lim_{t \to a+} \ \limsup_{k \to \infty} \gamma_{\alpha_i}(t, a + 1/k) = 0 \ \ (i \in \mathbb{N}_{1n_0}), \tag{5}$$

$$\lim_{t \to b-} b_i(t) d_1 a_i(t) < 1, \ \ \lim_{t \to b-} \ \limsup_{k \to \infty} \gamma_{\alpha_i}(t, b - 1/k) = 0 \ (i \in \mathbb{N}_{2n_0}), \tag{6}$$

where

$$\alpha_i(t) \equiv \int_{c_0}^t b_i(\tau)\, da_i(\tau), \quad c_{il}(t) \equiv \int_a^t \eta_{il}(\tau)\, da_i(\tau) \ \ (i, l = 1, \dots, n), \ \ c_0 = (a + b)/2.$$

Then the problem (1), (2) *is solvable.*

Corollary. *Let the conditions* (3)–(6) *hold, where the functions* a_i $(i = 1, \dots, n)$ *have no more than a finite number of points of discontinuity,* $b_i \in L_{loc}(I_i, \mathbb{R}_+; a_i)$, $\alpha_i(t) \equiv \int\limits_{c_0}^t b_i(\tau)\, da_i(\tau)$,

$$\int_a^t \eta_{il}(\tau)\, da_i(\tau) \equiv \int_c^t h_{il}(\tau)\, d\beta_l(\tau) \ \ (i, l = 1, \dots, n),$$

$c_0 = (a+b)/2$; β_l $(l = 1,\dots,n)$ *are nondecreasing functions,* $h_{ii} \in L^\mu([a,b],\mathbb{R};\beta_i)$, $h_{il} \in L^\mu([a,b],\mathbb{R}_+;\beta_l)$ $(i \neq l;\ i,l = 1,\dots,n)$; $1 \leq \mu \leq +\infty$. *Let, moreover,* $r(\mathcal{H}) < 1$, *where* $\mathcal{H} = (\mathcal{H}_{j+1\,m+1})_{j,m=0}^{2}$ *is the* $3n \times 3n$*-matrix defined by* $\mathcal{H}_{j+1\,m+1} = (\lambda_{kmij}\|h_{ik}\|_{\mu,s_m(\beta_i)})_{i,k=1}^{n}$ $(j,m = 0,1,2)$, $\xi_{ij} = (s_j(\beta_i)(b) - s_j(\beta_i)(a))^{\frac{1}{\nu}}$ $(j = 0,1,2;\ i = 1,\dots,n)$; $\lambda_{k0i0} = (\frac{4}{\pi^2})^{\frac{1}{\nu}}\xi_{k0}^2$ *if* $s_0(\beta_i)(t) \equiv s_0(\beta_k)(t)$, $\lambda_{k0i0} = \xi_{k0}\xi_{i0}$ *if* $s_0(\beta_i)(t) \not\equiv s_0(\beta_k)(t)$, $\lambda_{kmij} = \xi_{km}\xi_{ij}$ *if* $m^2 + j^2 > 0$ *and* $mj = 0$ $(j,m = 0,1,2)$, $\lambda_{kmij} = \big(\frac{1}{4}\,\mu_{\alpha_k m}\nu_{\alpha_k m \alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k m}+2}\big)^{\frac{1}{\nu}}$ $(j,m = 1,2)$ $(i,k = 1,\dots,n)$; $1/\mu + 2/\nu = 1$. *Then the problem* (1), (2) *is solvable.*

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References

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