On the General Nonlinear Boundary Value Problems for Systems of Discrete Equations

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There are considered the problem on the solvability of the system of nonlinear discrete equations

$$\Delta y(l-1)=g(l,y(l),y(l-1)) \ \text{ for } \ l\in\mathbb{N}_{m_0} \tag{1}$$

under the boundary value condition

$$\zeta(y)=0, \tag{2}$$

where $m_0\geq 2$ is a fixed natural number, $g\in \operatorname{Car}(\mathbb{N}_{m_0}\times\mathbb{R}^n,\mathbb{R}^n)$, and $\zeta: E(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)\to\mathbb{R}^n$ is a continuous, nonlinear in general, vector-functional.

We give the Conti–Opial type theorems (among them effective sufficient conditions) for the solvability of the problem which are analogous to ones given in [1] (see also the references therein) for ordinary differential equations.

Basic Notation and Definitions

$\mathbb{N}=\{1,2,\dots\}$, $\mathbb{N}_0=\{0,1,\dots\}$, $\mathbb{Z}$ is the set of all integers. If $m\in\mathbb{N}$, then $\mathbb{N}_m=\{1,\dots,m\}$, $\widetilde{\mathbb{N}}_m=\{0,1,\dots,m\}$. $\mathbb{R}=]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$. $\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$; $|X|=(|x_{ij}|)_{i,j}^{n,m}$. $\mathbb{R}_+^{n\times m}$ is the set of all real nonnegative $n\times m$-matrices. $I_{n\times n}$ is the identity $n\times n$-matrix. $r(X)$ is the spectral radius of $X\in\mathbb{R}^{n\times n}$. $\mathbb{R}^n=\mathbb{R}^{n\times 1}$.

$E(J,\mathbb{R}^{n\times m})$, where $J\subset\mathbb{Z}$, is the space of all matrix-functions $Y=(y_{ij})_{i,j=1}^{n,m}: J\to\mathbb{R}^{n\times m}$ with the norm $\|Y\|_J=\max\big\{\|Y(l)\|:\ l\in J\big\}$, $|Y|_J=(|y_{ij}|_J)_{i,j=1}^{n,m}$.

Δ is the difference operator of the first order, i.e. $\Delta Y(k-1)=Y(k)-Y(k-1)$ for $Y\in E(\widetilde{\mathbb{N}}_l,\mathbb{R}^{n\times m})$, $k\in\mathbb{N}_l$; If Y is defined on $\mathbb{N}_l$ or $\widetilde{\mathbb{N}}_{l-1}$, then we assume $Y(0)=O_{n\times m}$, or $Y(l)=O_{n\times m}$, respectively, if it is necessary.

If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ (nonlinear, in general) is said to be positive homogeneous if $g(\lambda x)=\lambda g(x)$ for every $\lambda\in R_+$ and $x\in B_1$; if, in addition, the spaces are partially ordered, then the operator g is called nondecreasing if $g(x)\leq g(y)$ for every $x,y\in B_1$ such that $x\leq y$.

If $J\subset\mathbb{Z}$, $D_1\subset\mathbb{R}^n$ and $D_2\subset\mathbb{R}^{n\times m}$, then $\operatorname{Car}(J\times D_1,D_2)$ is the discrete Carathéodory class, i.e., the set of all mappings $F:J\times D_1\to D_2$ such that the function $F(j,\cdot\,):D_1\to D_2$ is continuous for every $j\in J$.

By a solution of the difference problem (1), (2) we understand a vector-function $y\in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)$ satisfying both the system (1) for $i\in\{1,\dots,m_0\}$ and the boundary value condition (2).

Definition 1. Let $\mathcal{L}: E(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)\to\mathbb{R}^n$ be a linear continuous operator, and let $\mathcal{L}: E(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)\to\mathbb{R}_+^n$ be a positive homogeneous operator. We say that a pair (G_1,G_2), consisting of matrix-functions $G_j\in\operatorname{Car}(\mathbb{N}_{m_0}\times\mathbb{R}^{2n},\mathbb{R}^{n\times n})$ $(j=1,2)$, satisfy the Opial condition with respect to the pair $(\mathcal{L},\mathcal{L}_0)$ if:

(a) there exist a matrix-function $\Phi \in E(\mathbb{N}_{m_0}, \mathbb{R}^n_+)$ such that $|G_j(l,x,y)| \le \Phi(l)$ for $x, y \in \mathbb{R}^n$ $(j = 1, 2;\ l = 1, \dots, m_0)$;

(b) $\det\big(I_{n\times n} + (-1)^j B_j(l)\big) \neq 0$ $(j = 1, 2;\ l = 1, \dots, m_0)$ and the problem

$$\Delta y(l-1) = B_1(l)y(l) + B_2(l)y(l-1) \ \ (l \in \mathbb{N}_{m_0}), \ \ |\mathcal{L}(y)| \le \mathcal{L}_0(y) \tag{3}$$

has only the trivial solution for every matrix-functions $B_j \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$ for which there exists a sequences $x_k, y_k \in E(\mathbb{N}_{m_0}, \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$ such that

$$\lim_{k\to+\infty} G_j(l, x_k(l), y_k(l)) = B_j(l) \ \ (j = 1, 2;\ \ l = 1, \dots, m_0).$$

Theorem 1. *Let the conditions*

$$\big\|g(l,x,y) - G_1(l,x,y)x - G_2(l,x,y)y\big\| \le \alpha\big(l, \|x\| + \|y\|\big) \ \ \textit{for}\ l \in \mathbb{N}_{m_0}, \ \ x, y \in \mathbb{R}^n, \tag{4}$$

$$\big|\zeta(y) - \mathcal{L}(y)\big| \le \mathcal{L}_0(y) + \ell_1\big(\|y\|_{m_0}\big) \ \ \textit{for}\ \ y \in E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n) \tag{5}$$

hold, where $\mathcal{L} : E(\widetilde{E}_{m_0}, \mathbb{R}^n) \to \mathbb{R}^n$ and $\mathcal{L}_0 : E(\widetilde{E}_{m_0}, \mathbb{R}^n) \to \mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators, the pair (G_1, G_2) satisfies the Opial condition with respect to the pair $(\mathcal{L}, \mathcal{L}_0)$; $\alpha \in \mathrm{Car}(\widetilde{E}_{m_0} \times \mathbb{R}_+, \mathbb{R}_+)$ is a function nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ is a nondecreasing vector-function such that

$$\limsup_{\rho\to+\infty} \frac{1}{\rho} \Big(\|\ell_1(\rho)\| + \sum_{l=1}^{m_0} \alpha(l, \rho)\Big) < 1. \tag{6}$$

Then the problem (1), (2) *is solvable.*

Theorem 2. *Let the conditions* (4)–(6) *and $P_{j1}(l) \le G_j(l,x,y) \le P_{j2}(l)$ for $l \in \mathbb{N}_{m_0}$, $x, y \in \mathbb{R}^n$ $(j=1,2)$ hold, where $P_{j1}, P_{j2} \in E(\mathbb{N}_{m_0}, \mathbb{R}^n)$ $(j=1,2)$, $\mathcal{L} : E(\widetilde{E}_{m_0}, \mathbb{R}^n) \to \mathbb{R}^n$ and $\mathcal{L}_0 : E(\widetilde{E}_{m_0}, \mathbb{R}^n) \to \mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators; and $\alpha \in \mathrm{Car}(\widetilde{E}_{m_0} \times \mathbb{R}_+, \mathbb{R}_+)$ is a function nondecreasing in the second variable and $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ is a nondecreasing vector-function. Let, moreover, the inequalities in* (c) *of Definition* 1 *and the problem* (3) *have only the trivial solution for every matrix-functions B_1 and B_2 from $E(\mathbb{N}_{m_0}, \mathbb{R}^n)$ such that $P_{j1}(l) \le B_j(l) \le P_{j2}(l)$ for $l \in \mathbb{N}_{m_0}$ $(j = 1, 2)$. Then the problem* (1), (2) *is solvable.*

Remark. Theorem 2 is interesting only in the case when $G_j(l, \cdot, \cdot) \notin C(\mathbb{R}^{2n}, \mathbb{R}^{n\times n})$ for some $j \in \{1, 2\}$ and $l \in \{1, \dots, m_0\}$, because it immediately follows from Theorem 1 in the case when $G_j \in \mathrm{Car}(\mathbb{N}_{m_0} \times \mathbb{R}^{2n}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$.

Corollary 1. *Let the conditions* (4)–(6) *hold, where $G_j(l,x,y) \equiv G_j(l)$ $(j = 1, 2)$; $G_1, G_2, \in E(\mathbb{N}_{m_0}, \mathbb{R}^n)$; $\mathcal{L}_0(y) \equiv 0$, $\mathcal{L} : E(\widetilde{E}_{m_0}, \mathbb{R}^n) \to \mathbb{R}^n$ is a linear continuous operator; $\alpha \in \mathrm{Car}(\widetilde{E}_{m_0} \times \mathbb{R}_+, \mathbb{R}_+)$ is a function nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ is a nondecreasing function. Let, moreover,* $\det\big(I_{n\times n} + (-1)^j G_j(l)\big) \neq 0$ *for $l \in \mathbb{N}_{m_0}$ $(j = 1, 2)$ and the problem $\Delta y(l-1) = G_1(l)y(l) + G_2(l)y(l-1)$ $(l \in \mathbb{N}_{m_0})$, $\mathcal{L}(y) = 0$ have only the trivial solution. Then the problem* (1), (2) *is solvable.*

We give the effective conditions for the solvability of the problem (1), (2).

On the set $E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times n}) \times E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times})$ we introduce the operators by the following way. If $G_1, G_2 \in E(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times n})$ and, in addition, $\det(I_{n\times n} + G_2(l)) \neq 0$ $(l = 1, \dots, m_0)$, then we assume

$$\big[(G_1, G_2)(l)\big]_0 \equiv I_n, \ \ \big[(G_1, G_2)(l)\big]_k \equiv -\sum_{i=l+1}^{m_0} \big(G_1(i) + G_2(i+1)\big) \times$$
$$\times \big(I_n + G_2(i)\big)^{-1} \big[(G_1, G_2)(i)\big]_{k-1} \ \ (k = 1, 2, \dots);$$

$$V_1(G_1,G_2)(l) \equiv \sum_{i=l+1}^{m_0} \Big|\big(G_1(i)+G_2(i+1)\big)\big(I_n+G_2(i+1)\big)^{-1}\Big|,$$

$$V_{k+1}(G_1,G_2)(l) \equiv \sum_{i=l+1}^{m_0} \Big|\big(G_1(i)+G_2(i+1)\big)\big(I_n+G_2(i+1)\big)^{-1}\Big| \cdot V_k(G_1,G_2)(i) \ \ (k=1,2,\dots).$$

Theorem 3. *Let the conditions* (4)–(6) *hold, where* $G_j(l,x,y) \equiv G_j(l)$ $(j=1,2)$*;* $G_1, G_2, \in E(\mathbb{N}_{m_0}, \mathbb{R}^n)$*;* $\alpha \in \mathrm{Car}(\widetilde{E}_{m_0} \times \mathbb{R}_+, \mathbb{R}_+)$ *is a function nondecreasing in the second variable,* $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ *is a nondecreasing function;* $\mathcal{L}_0(y) \equiv 0$*;* $\mathcal{L}(y) \equiv \sum\limits_{j=1}^{n_0} L_j y(k_j)$*,* n_0 *is a natural number,* $k_j \in \widetilde{\mathbb{N}}_{m_0}$ *and* $L_j \in \mathbb{R}^{n\times n}$ $(j=1,\dots,n_0)$*. Let, moreover,* $\det\big(I_{n\times n}+(-1)^j G_j(l)\big) \neq 0$ *for* $l \in \mathbb{N}_{m_0}$ $(j=1,2)$ *and there exist natural numbers* k *and* m *such that* $\det(M_k) \neq 0$ *and* $r(M_{k,m}) < 1$*, where*

$$M_k = \sum_{j=1}^{n_0} \sum_{i=0}^{k-1} L_j \big(I_n + G_2(k_j+1)\big)^{-1} \big[(G_1,G_2)(k_j)\big]_i, \quad M_{k,m} = V_m(G_1,G_2)(0) +$$

$$+ \sum_{i=0}^{m-1} \big| \big[(G_1,G_2)(\,\cdot\,)\big]_i \big|_{\widetilde{\mathbb{N}}_{m_0}} \cdot \sum_{j=1}^{n_0} |M_k^{-1} L_j| \big(I_n + G_2(k_j+1)\big)^{-1} V_k(G_1,G_2)(k_j).$$

Then the problem (1), (2) *is solvable.*

Corollary 2. *Let the conditions* (4)–(6) *hold, where* $G_j(l,x,y) \equiv G_j(l)$ $(j=1,2)$*;* $G_1, G_2, \in E(\mathbb{N}_{m_0}, \mathbb{R}^n)$*,* $\det\big(I_{n\times n}+(-1)^j G_j(l)\big) \neq 0$ *for* $l \in \mathbb{N}_{m_0}$ $(j=1,2)$*;* $\alpha \in \mathrm{Car}(\widetilde{E}_{m_0} \times \mathbb{R}_+, \mathbb{R}_+)$ *is a function nondecreasing in the second variable,* $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ *is a nondecreasing function;* $\mathcal{L}_0(y) \equiv 0$*;* $\mathcal{L}(y) \equiv \sum\limits_{j=1}^{n_0} L_j y(k_j)$*,* n_0 *is a natural number,* $k_j \in \widetilde{\mathbb{N}}_{m_0}$ *and* $L_j \in \mathbb{R}^{n\times n}$ $(j=1,\dots,n_0)$*. Let, moreover,*

$$\det\Big(\sum_{j=1}^{n_0} L_j \big(I_n + G_2(k_j+1)\big)^{-1}\Big) \neq 0$$

and $r(L_0 M_0) < 1$*, where*

$$L_0 = I_n + \Big| \Big(\sum_{j=1}^{n_0} L_{1j} \big(I_n + G_2(k_j+1)\big)^{-1} \Big)^{-1} \Big| \cdot \sum_{j=1}^{n_0} \big| L_j \big(I_n + G_1(k_j)\big)^{-1} \big|;$$

$$M_0 = \sum_{i=1}^{m_0} \Big|\big(G_1(i)+G_2(i+1)\big)\big(I_n+G_2(i+1)\big)^{-1}\Big|.$$

Then the problem (1), (2) *is solvable.*

Acknowledgement

This work is supported by the Shota Rustaveli National Science Foundation (Grant No. FR/182/5-101/11).

References

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