On the Nonlinear Boundary Value Problems for Systems of Impulsive Equations with Finite Points of Impulses Actions

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We consider the general nonlinear boundary value problem for the system of impulsive equations with finite number of impulses points

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \tag{1}$$

$$x(\tau_l+) - x(\tau_l-) = I_l(x(\tau_l)) \ \ (l = 1,\dots,m_0); \tag{2}$$

$$h(x) = 0, \tag{3}$$

where $-\infty < a < \tau_1 < \cdot < \tau_{m_0} \leq b < \infty$, m_0 is a natural number, $f \in \mathrm{Car}([a,b] \times \mathbb{R}^n, \mathbb{R}^n)$, $I_l : R^n \to \mathbb{R}^n$ $(l = 1,\dots,m_0)$ are continuous operators, and $h : C_s([a,b],\mathbb{R}^n;m_0) \to \mathbb{R}^n$ is a continuous, nonlinear in general, vector-functional.

We give the Conti–Opial type theorems (among them effective sufficient conditions) for the solvability of the problem which are analogous to ones given in [1] (see also the references therein) for ordinary differential equations.

Basic notation and definitions. $\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$. $\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$; $|X| = (|x_{ij}|)_{i,j}^{n,m}$. $\mathbb{R}_+^{n\times m}$ is the set of all real nonnegative $n \times m$-matrices. $I_{n\times n}$ is the identity $n \times n$-matrix. $r(X)$ is the spectral radius of $X \in \mathbb{R}^{n\times n}$. $\mathbb{R}^n = \mathbb{R}^{n\times 1}$.

$X(t-)$ and $X(t+)$ are one-sided limits of the matrix-function X at the point t.

$C([a,b],\mathbb{R}^n;m_0)$ is the set of all vector-functions $x : [a,b] \to \mathbb{R}^n$, having the one sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1,\dots,m_0)$, whose restrictions to every closed interval from $[a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}$ is continuous. $C_s([a,b],\mathbb{R}^n;m_0)$ is the Banach space with the norm $\|x\|_s = \sup\{\|x(t)\| : t \in [a,b]\}$.

$\widetilde{C}([a,b],\mathbb{R}^n;m_0)$ is the set of all matrix-functions $x : [a,b] \to \mathbb{R}^n$, having the one sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1,\dots,m_0)$, whose restrictions to an arbitrary closed interval from $[a,b] \setminus \{\tau_k\}_{k=1}^{m}$ is absolutely continuous.

If B_1 and B_2 are normed spaces, then an operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x) = \lambda g(x)$ for every $\lambda \in R_+$ and $x \in B_1$. An operator $\varphi : C([a,b],\mathbb{R}^{n\times m};m_0) \to R^n$ is called nondecreasing if for every x,y such that $x(t) \leq y(t)$ for $t \in [a,b]$ the inequality $\varphi(x)(t) \leq \varphi(y)(t)$ holds for $t \in [a,b]$.

$\mathrm{Car}([a,b] \times D_1, D_2)$, where $D_1 \subset R^n$ and $D_2 \subset R^{n\times m}$, is the standard Carathéodory class of all mappings $F : [a,b] \times D_1 \to D_2$; $\mathrm{Car}^0([a,b] \times D_1, D_2)$ is the set of all mappings F such that the matrix-function $F(.,x(.))$ is measurable for every vector-function $x \in C([a,b],D_1;m_0)$.

By a solution of the impulsive system $(1),(2)$ we understand a continuous from the left vector-function $x \in \widetilde{C}([a,b],\mathbb{R}^n;m_0)$ satisfying both the system (1) a. e. on $[a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}$ and the relation (2) for every $k \in \{1,\dots,m_0\}$.

Definition. Let $\ell : C_s([a,b],\mathbb{R}^n;m_0) \to \mathbb{R}^n$ and $\ell_0 : C_s([a,b],\mathbb{R}^n;m_0) \to \mathbb{R}^n_+$ be, respectively, a linear continuous and a positive homogeneous operators. We say that a pair $(P,\{J_l\}_{l=1}^{m_0})$, consisting of a matrix-function $P \in \operatorname{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and a finite sequence of continuous operators $J_l=(J_{li})_{i=1}^n : \mathbb{R}^n \to \mathbb{R}^n$ $(l=1,\dots,m_0)$, satisfy the Opial condition with respect to the pair (ℓ,ℓ_0) if:

(a) there exist a matrix-function $\Phi \in L([a,b],\mathbb{R}^n_+)$ and constant matrices $\Psi_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ such that $|P(t,x)| \le \Phi(t)$ for a.e. $[a,b]$ and $x\in\mathbb{R}^n$;

(b) $|J_l(x)| \le \Psi_l$ for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$;

(c) $\det(I_{n\times n}+G_l)\neq 0$ $(l=1,\dots,m_0)$ and the problem

$$\frac{dx}{dt} = A(t)x,\ \ x(\tau_l+)-x(\tau_l-) = G_l x(\tau_l)\ \ (l=1,\dots,m_0);\ \ |\ell(x)| \le \ell_0(x) \tag{4}$$

has only the trivial solution for every matrix-function $A\in L([a,b],\mathbb{R}^{n\times n})$ and constant matrices $G_1,\dots,G_{m_0}$ for which there exists a sequence $y_k\in\widetilde{C}([a,b],\mathbb{R}^n;m_0)$ $(k=1,2,\dots)$ such that $\lim\limits_{k\to+\infty}\int_a^t (P(\tau,y_k(\tau))-A(\tau))\,d\tau=0$ uniformly on $[a,b]$ and $\lim\limits_{k\to+\infty} J_l(y_k(\tau_l))=G_l$ $(l=1,\dots,m_0)$.

Theorem 1. *Let the conditions*

$$\|f(t,x)-P(t,x)x\| \le q\big(t,\|x\|\big)\ \ \text{a.e. on}\ \ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\},\ \ x\in\mathbb{R}^n, \tag{5}$$

$$\|I_l(x)-J_l(x)x\| \le \beta_l(\|x\|)\ \ \text{for}\ \ x\in\mathbb{R}^n\ \ (l=1,\dots,m_0), \tag{6}$$

$$|h(x)-\ell(x)| \le \ell_0(x)+\ell_1(\|x\|_s)\ \ \text{for}\ \ x\in \operatorname{BV}([a,b],\mathbb{R}^n) \tag{7}$$

hold, where $\ell : C_s([a,b],\mathbb{R}^n;m_0)\to\mathbb{R}^n$ and $\ell_0 : C_s([a,b],\mathbb{R}^n;m_0)\to\mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators, the pair $(P,\{J_l\}_{l=1}^{m_0})$ satisfies the Opial condition with respect to the pair (ℓ,ℓ_0); $q\in\operatorname{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is a function nondecreasing in the second variable, and $\beta_l\in C(\mathbb{R}_+,\mathbb{R}_+)$ $(l=1,\dots,m_0)$ and $\ell_1\in C(\mathbb{R}_+,\mathbb{R}^n_+)$ are nondecreasing, respectively, functions and vector-function such that

$$\limsup_{\rho\to+\infty}\frac{1}{\rho}\bigg(\|\ell_1(\rho)\| + \int_a^b q(t,\rho)\,dt + \sum_{l=1}^{m_0}\beta_l(\rho)\bigg) < 1. \tag{8}$$

Then the problem (1), (2); (3) *is solvable.*

Theorem 2. *Let the conditions* (5)–(8), *$P_1(t)\le P(t,x)\le P_2(t)$ for a.a. $t\in[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}$ and $x\in\mathbb{R}^n$, and $J_{1l}\le I_k(x)\le J_{2l}$ for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$ hold, where $P\in\operatorname{Car}^0([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i\in L([a,b],\mathbb{R}^{n\times n})$ $(i=1,2)$, $J_{il}\in\mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$, $\ell : C_s([a,b],\mathbb{R}^n;m_0)\to\mathbb{R}^n$ and $\ell_0 : C_s([a,b],\mathbb{R}^n;m_0)\to\mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators; $q\in\operatorname{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is a function nondecreasing in the second variable, and $\beta_l\in C([a,b],\mathbb{R}_+)$ $(l=1,\dots,m_0)$ and $\ell_1\in C(\mathbb{R}_+,\mathbb{R}^n_+)$ are nondecreasing, respectively, functions and vector-function. Let, moreover, the inequalities in* (c) *of definition hold and the problem* (4) *have only the trivial solution for every matrix-function $A\in L([a,b],\mathbb{R}^{n\times n})$ and constant matrices $G_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ such that $P_1(t)\le A(t)\le P_2(t)$ for a.a. $t\in[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}$ and $x\in\mathbb{R}^n$, and $J_{1l}\le G_l\le J_{2l}$ for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$. Then the problem* (1), (2); (3) *is solvable.*

Remark. Theorem 2 is interesting only in the case when $P\notin\operatorname{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, because otherwise it follows from Theorem 1.

Corollary 1. *Let the conditions* (5)–(8) *hold, where $P(t,x)\equiv P(t)$, $P\in L([a,b],\mathbb{R}^{n\times n})$, $J_l(x)\equiv J_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ are constant matrices, $\ell_0(x)\equiv 0$, $\ell : C_s([a,b],\mathbb{R}^n;m_0)\to\mathbb{R}^n$ is a linear continuous operator, $q\in\operatorname{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is a function nondecreasing in the second variable, and $\beta_l\in C([a,b],\mathbb{R}_+)$ $(l=1,\dots,m_0)$ and $\ell_1\in C(\mathbb{R}_+,\mathbb{R}^n_+)$ are nondecreasing, respectively, functions and vector-function. Let, moreover, $\det(I_{n\times n}+J_l)\neq 0$ $(l=1,\dots,m_0)$ and the homogeneous impulsive problem $dx/dt=P(t)x$, $x(\tau_l+)-x(\tau_l-)=J_l x(\tau_l)$ $(l=1,\dots,m_0)$; $\ell(x)=0$ have only the trivial solution. Then the problem* (1), (2); (3) *is solvable.*

We give the effective conditions for the solvability of the problem (1), (2); (3).

For every matrix-function $X \in L([a,b],\mathbb{R}^{n\times n})$ and constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k=1,\dots,m_0)$ we introduce the operators $\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_0 \equiv I_{n\times n}$, $\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_{i+1} \equiv \int\limits_a^t X(\tau)\big[(X,Y_1,\dots,Y_{m_0})(\tau)\big]_i\,d\tau + \sum\limits_{a\le\tau_l<t} Y_l\big[(X,Y_1,\dots,Y_{m_0})(\tau_l)\big]_i$ $(i=0,1,\dots)$.

Corollary 2. *Let the matrix-function $P \in L([a,b],\mathbb{R}^{n\times n})$, constant matrices $J_l \in \mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$, the functions $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$, $\beta_l \in C(\mathbb{R}_+,\mathbb{R}_+)$ $(l=1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R}_+,\mathbb{R}_+)$ satisfy the conditions of Corollary 1, where $\ell_0(x)\equiv 0$, and $\ell(x) \equiv \int\limits_a^b d\mathcal{L}(t)\cdot x(t)$, $\mathcal{L} \in L([a,b],\mathbb{R}^{n\times n})$. Let, moreover, there exist natural numbers k and m such that the matrix $M_k = -\sum\limits_{i=0}^{k-1}\int\limits_a^b d\mathcal{L}(t)\cdot\big[(P,J_l,\dots,J_{m_0})(t)\big]_i$ is nonsingular and $r(M_{k,m})<1$, where $M_{k,m} = \big[(|P|,|J_1|,\dots,|J_{m_0}|)(b)\big]_m + \sum\limits_{i=0}^{m-1}\big[(|P|,|J_1|,\dots,|J_{m_0}|)(b)\big]_i\int\limits_a^b dV(M_k^{-1}\mathcal{L})(t)\cdot\big[(|P|,|J_1|,\dots,|J_{m_0}|)(t)\big]_k$. Then the problem (1), (2); (3) is solvable.*

Corollary 3. *Let the matrix-function $P \in L([a,b],\mathbb{R}^{n\times n})$, constant matrices $J_l \in \mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$, the functions $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$, $\beta_l \in C(\mathbb{R}_+,\mathbb{R}_+)$ $(l=1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R}_+,\mathbb{R}_+)$ satisfy the conditions of Corollary 1, where $\ell_0(x)\equiv 0$, and $\ell(x)\equiv\sum\limits_{j=1}^{n_0}\mathcal{L}_j x(t_j)$, $t_j\in[a,b]$, $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j=1,\dots,n_0)$. Let, moreover, there exist natural numbers k and m such that the matrix $M_k = \sum\limits_{j=1}^{n_0}\sum\limits_{i=0}^{k-1}\mathcal{L}_j\big[(P_0,J_l,\dots,J_{m_0})(t_j)\big]_i$ is nonsingular and $r(M_{k,m})<1$, where $M_{k,m} = \big[(|P|,|J_l|,\dots,|J_{m_0}|)(b)\big]_m + \Big(\sum\limits_{i=0}^{m-1}\big[(|P|,|J_l|,\dots,|J_{m_0}|)(b)\big]_i\Big)\cdot\sum\limits_{j=1}^{n_0}|M_k^{-1}\mathcal{L}_j|\Big[(|P|,|J_l|,\dots,|J_{m_0}|)(t_j)\Big]_k$. Then the problem (1), (2); (3) is solvable.*

Corollary 4. *Let the matrix-function $P \in L([a,b],\mathbb{R}^{n\times n})$, constant matrices $J_l \in \mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$, the functions $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$, $\beta_l \in C(\mathbb{R}_+,\mathbb{R}_+)$ $(l=1,\dots,m_0)$ and $\ell_1 \in C(\mathbb{R}_+,\mathbb{R}_+)$ satisfy the conditions of Corollary 1, where $\ell_0(x)\equiv 0$, and $\ell(x)\equiv\sum\limits_{j=1}^{n_0}\mathcal{L}_j x(t_j)$, $t_j\in[a,b]$, $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j=1,\dots,n_0)$. Let, moreover, $\det\Big(\sum\limits_{j=1}^{n_0}\mathcal{L}_j\Big)\neq 0$ and $r\big(\mathcal{L}_0\cdot V(A)(b)\big)<1$, where $\mathcal{L}_0 = I_{n\times n} + \Big|\Big(\sum\limits_{j=1}^{n_0}\mathcal{L}_j\Big)^{-1}\Big|\cdot\sum\limits_{j=1}^{n_0}|\mathcal{L}_j|$ and $A = \int\limits_a^b |P(t)|\,dt + \sum\limits_{l=1}^{m_0}|J_l|$. Then the problem (1), (2); (3) is solvable.*

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References

[1] I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) Translated in *J. Soviet Math.* **43** (1988), No. 2, 2259–2339. Itogi Nauki i Tekhniki, *Current problems in mathematics. Newest results, Vol. 30 (Russian)*, 3–103, 204, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1987.