Feedback Stabilization of Complex System Behavior

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1 Introduction

The famous Hodgkin–Huxley neuron model [4] is the first mathematical model describing neural excitation transmission derived from the angle of physics and lays the basis of electrical neurophysiology. FitzHugh–Nagumo equation [3, 7], which is a simplification of Hodgkin-Huxley model, describes the generation and propagation of the nerve impulse along the giant axon of the squid. Based on the finite propagating speed in the signal transmission between the neurons, the following coupled FitzHugh–Nagumo neural system is proposed [5]:

$$\left|\begin{aligned} &\dot{u}_1 = -u_1(u_1-1)(u_1-a) - u_2 + cf(u_3),\\ &\dot{u}_2 = b(u_1 - \gamma u_2),\\ &\dot{u}_3 = -u_3(u_3-1)(u_3-a) - u_4 + cf(u_1),\\ &\dot{u}_4 = b(u_3 - \gamma u_4), \end{aligned}\right. \tag{1}$$

where a, b, γ are positive constants, $u_{1,2}$ represent transmission variables, and $u_{3,4}$ are receiving variables; c measures the coupling strength, $f \in C^3$, $f(0) = 0$, $f'(0) = 1$. We shall take $f(x) = \tan h(x)$ in our investigation. System (1) is symmetric.

2 Edge of Chaos of Coupled FitzHugh–Nagumo CNN Model

We apply the following constructive algorithm [1, 2] for studying the dynamics of (1):

1. Map coupled FitzHugh–Nagumo system (1) into the associated discrete- space version, which we shall call coupled FitzHugh–Nagumo Cellular Neural Networks (CNN) model [6]:

$$\left|\begin{aligned} &\frac{du_j^1}{dt} = -u_j^1(u_j^1-1)(u_j^1-a) - u_j^2 + cf(u_j^3),\\ &\frac{du_j^2}{dt} = b(u_j^1 - \gamma u_j^2),\\ &\frac{du_j^3}{dt} = -u_j^3(u_j^3-1)(u_j^3-a) - u_j^4 + cf(u_j^1),\\ &\frac{du_j^4}{dt} = b(u_j^3 - \gamma u_j^4), \quad j = 1, \dots, n. \end{aligned}\right. \tag{2}$$

The system is transformed into a system of ordinary differential equations which is identified as the state equations of a CNN with appropriate templates. We map the variables u_1, u_2, u_3 and u_4 into CNN layers [6] such that the state voltage of a CNN cell at a grid point is $u_j^i, i = 1, 2, 3, 4$, $n = M.M$, M is number of the cells.

2. Find the equilibrium points of (2). According to the theory of dynamical systems the equilibrium points $\widehat{u}_j^i$ of (2) are these for which:

$$\left| \begin{array}{l} -u_j^1(u_j^1-1)(u_j^1-a)-u_j^2+c\tan h(u_j^3)=0, \\ \ b(u_j^1-\gamma u_j^2)=0, \\ -u_j^3(u_j^3-1)(u_j^3-a)-u_j^4+c\tan h(u_j^1)=0, \\ \ b(u_j^3-\gamma u_j^4)=0. \end{array} \right. \tag{3}$$

Equation (3) may have one, two,three or four real roots $\widehat{u}_j^1$, $\widehat{u}_j^2$, $\widehat{u}_j^3$, $\widehat{u}_j^4$ respectively. In general, these roots are functions of the cell parameters a, b, c, γ. In other words, we have $\widehat{u}_j^i = \widehat{u}_j^i(a, b, c, \gamma)$, $i = 1, 2, 3, 4$. We shall consider only the equilibrium point $E_0 = (0, 0, 0, 0)$.

3. Calculate now the Jacobian matrix of (3) about equilibrium point E_0. In our particular case the associate linear system in a sufficient small neighborhood of the equilibrium point E_0 can be given by

$$\frac{dz}{dt} = DF(E_0)z$$

$DF(E_0) = J$ is the Jacobian matrix of each of the equilibrium points and can be computed by:

$$J_{p,s} = \left.\frac{\partial F_p}{\partial u_s}\right|_{u=E_0}, \quad 1 \le p, s \le n. \tag{4}$$

4. Calculate the trace $\mathrm{Tr}(E_0) = \sum_{q=1}^{N} \lambda_q$. In the equilibrium point $E_0 = (0, 0, 0, 0)$ trace is $\mathrm{Tr}(0, 0, 0, 0) = -a - b\gamma - a - b\gamma = -2(a + b\gamma)$.

Definition 1. Stable and Locally Active Region SLAR(E) at the equilibrium point E_0 for coupled FitzHugh–Nagumo CNN model (2) is such that $\mathrm{Tr} < 0$.

In our particular case we have: $\mathrm{Tr}(0, 0, 0, 0) = -2(a + b\gamma) < 0$ for all a, b, γ positive. Therefore in the equilibrium point $E_0 = (0, 0, 0, 0)$ we have stable and locally active region.

5. Edge of chaos.

We shall identify the edge of chaos domain (EC) in the cell parameter space by using the following definition [1, 2]:

Definition 2. Coupled FitzHugh–Nagumo CNN model is said to be operating on the edge of chaos EC iff there is at least one equilibrium point E_0, which belongs to SLAR(E).

The following theorem then holds:

Theorem 1. *CNN model of coupled FitzHugh–Nagumo system* (1) *is operating in the edge of chaos regime for all a, b and γ positive. For this parameter values there is at least one equilibrium point which belongs to* SLAR(E).

3 Stabilizing Feedback Control for Coupled FitzHugh–Nagumo CNN Model

Let us extend the model (2) by adding to each cell the local linear feedback:

$$\left|\begin{aligned} \frac{du_j^1}{dt} &= -u_j^1(u_j^1-1)(u_j^1-a) - u_j^2 + cf(u_j^3) - ku_j^1, \\ \frac{du_j^2}{dt} &= b(u_j^1 - \gamma u_j^2), \\ \frac{du_j^3}{dt} &= -u_j^3(u_j^3-1)(u_j^3-a) - u_j^4 + cf(u_j^1) - ku_j^3, \\ \frac{du_j^4}{dt} &= b(u_j^3 - \gamma u_j^4), \quad j = 1, \ldots, n, \end{aligned}\right. \tag{5}$$

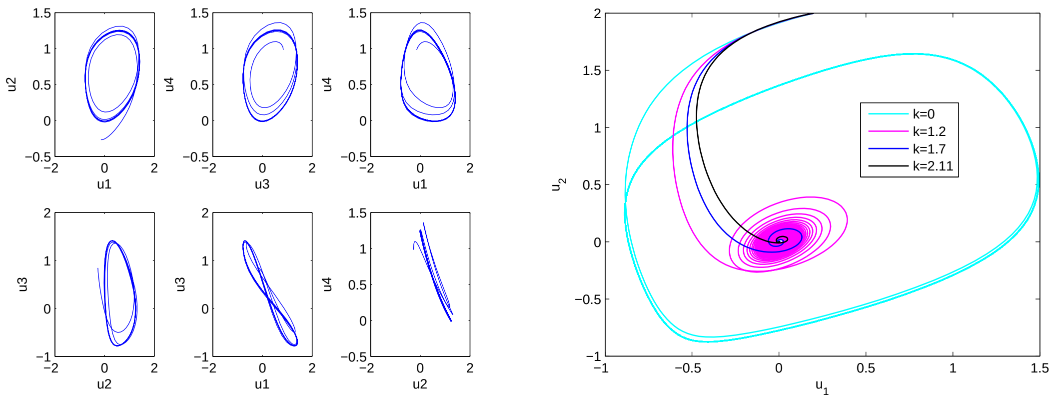
where k is the feedback controls coefficient, which is assumed to be equal for all cells. The problem is to prove that this simple and available for the implementation feedback can stabilize the coupled FitzHugh–Nagumo CNN model. In the following we present a proof of this statement and give sufficient condition on the feedback coefficient values which provide stability of the CNN nonlinear model (5).

As a first step, we examine the the stability conditions of the system (5), linearized in the neighborhood of the zero equilibrium point E_0. This system in a vector-matrix form is given by

$$\frac{dz}{dt} = J(k)z,$$

$J(k)$ is the Jacobian matrix of the controlled CNN in E_0:

$$J(k) = \begin{bmatrix} -(a+k) & -1 & c & 0 \\ b & -b\gamma & 0 & 0 \\ c & 0 & -(a+k) & -1 \\ 0 & 0 & b & -b\gamma \end{bmatrix}. \tag{6}$$



(a) (b)

Figure 1. (a) EC phenomena for CNN model (2); (b) Phase trajectory $u_1 - u_2$ for different values on the feedback coefficient CNN model (5).

Theorem 2. *Let the parameters a, b and γ of coupled FitzHugh–Nagumo CNN system and feedback coefficient k (5) have positive values. Then its linearized in E_0 model (6) is asymptotically stable for all*

$$k > \sqrt{\left(\frac{(b-1)^2}{8b\gamma}\right)^2 + c^2} + \frac{(b-1)^2}{8b\gamma} - a. \tag{7}$$

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References

[1] L. O. Chua, Local activity is the origin of complexity. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **15** (2005), No. 11, 3435–3456.

[2] R. Dogaru and L. O. Chua, Edge of chaos and local activity domain of FitzHugh–Nagumo equation. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **8** (1998), No. 2, 211–257.

[3] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane. *Biophysical J.* **1** (1961), No. 6, 445–466.

[4] A. L. Hodgkin, The local electric changes associated with repetitive action in a non-medullated axon. *J. Physiology* **107** (1948), No. 2, 165–181.

[5] J. Wu, Symmetric functional-differential equations and neural networks with memory. *Trans. Amer. Math. Soc.* **350** (1998), No. 12, 4799–4838.

[6] A. Slavova, Applications of some mathematical methods in the analysis of cellular neural networks. *J. Comput. Appl. Math.* **114** (2000), No. 2, 387–404.

[7] A. Slavova and P. Zecca, CNN model for studying dynamics and travelling wave solutions of FitzHugh–Nagumo equation. *J. Comput. Appl. Math.* **151** (2003), No. 1, 13–24.