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ON THE CONSTRUCTION OF SOLUTIONS OF CERTAIN SPATIAL AXISYMMETRIC MIXED PROBLEMS OF FILTRATION WITH PARTIALLY UNKNOWN BOUNDARIES

The axis of symmetry is assumed to be the x-axis directed downwards, the distance to the x-axis is denoted by y, and the velocity vector is expressed as follows: $\vec{V}(u,v) = \operatorname{grad}\varphi$. The conditions for incompressibility and potentiality of a moving liquid are of the form $\operatorname{div}(y\vec{V}) = 0$ (1), $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$ (2). The streamline equation $vdx - udy = 0$ multiplied by y becomes the exact differential of the stream function $\psi(x,y)$, where $u = \frac{\partial \varphi}{\partial x} = y^{-1}\frac{\partial \psi}{\partial y}$ (3), $v = \frac{\partial \varphi}{\partial y} = -y^{-1}\frac{\partial \psi}{\partial x}$ (4). The functions $\varphi(x,y)$ and $\psi(x,y)$ with respect to y are even functions (surface of rotation). Therefore $y^{-1}\frac{\partial \varphi}{\partial y}$ and $y^{-1}\frac{\partial \psi}{\partial y}$ tend to finite limits, as $y \to \infty$. The domain $S(z)$ with the boundary $\ell(z)$, occupied by a moving liquid we combine with the plane $z = x + iy$. The boundary $\ell(z)$ consists of an unknown curve and known segments, lines and of their portions. We seek for the functions $\omega(z) = \varphi(x,y) + i\psi(x,y)$, $w(z) = u(x,y) + iv(x,y)$, where $\varphi(x,y)$ and $\psi(x,y)$ must satisfy equations (3),(4) and also the following boundary conditions: $\psi(x,y) = \mathit{const}$ along nonpermeable boundaries; $\varphi(x,y) = \mathit{const}$ along water boundaries; $\varphi(x,y) - kx = \mathit{const}$, $\psi(x,y) = Q$, $k = \mathit{const}$, $Q = \mathit{const}$ along an unknown curve; $\varphi(x,y) - kx = \mathit{const}$ along the leaking interval; $\psi(x,y) = 0$, $(x,y) \in \ell(z)$, along the axis of symmetry. To the above-mentioned conditions are added to the corresponding equations of curves.

Let us consider the right half of the plane axisymmetric domain $S_0(z)$ with the boundary $\ell_0(z)$ which coincides with the domain $S(z)$ with the boundary $\ell(z)$ and respectively with the boundary conditions. The equation of depression curve in the plane problem must be the function y^2. Assume that to the domain $S_0(z)$ there correspond the domains $S(\omega_0)$ and $S(w_0)$ of a complex potential $\omega_0(z) = \varphi_0(x,y) + i\psi_0(x,y)$ and of a complex velocity $w_0(z) = \frac{d\omega_0(z)}{dz}$. Geometrical characteristics respectively of the domains $S(z) = S(z_0)$, $S(\omega) = S(\omega_0)$, $S(w) = S(w_0)$ and boundaries $\ell(z) = \ell(z_0)$, $\ell(\omega) = \ell(\omega_0)$, $\ell(w) = \ell(w_0)$ coincide. The boundary conditions are likewise coincide. Here we emphasize that the functions $\omega_0(z)$ and $w_0(z)$ are holomorphic, while the functions $\omega(z)$ and $w(z)$ are analytic ones.

The half-plane $\operatorname{Im}(\zeta) > 0$ of the plane $\zeta = \xi + i\eta$ is conformally mapped onto the domains $S(z_0)$, $S(\omega_0)$ and $S(w_0)$. Conformally mapping functions we denote by $z_0(\zeta)$, $\omega_0(\zeta)$ and $w_0(\zeta)$. Generalized functions $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ must satisfy the equations $\frac{\partial \varphi}{\partial \xi} = y^{-1}\frac{\partial \psi}{\partial \eta}$ (5), $\frac{\partial \varphi}{\partial \eta} = y^{-1}\frac{\partial \psi}{\partial \xi}$ (6), or $\Delta\varphi(\xi,\eta) + y^{-1}(\frac{\partial y}{\partial \xi}\frac{\partial \varphi}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial \varphi}{\partial \eta}) = 0$ (7), $\Delta\psi(\xi,\eta) - y^{-1}(\frac{\partial y}{\partial \xi}\frac{\partial \psi}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial \psi}{\partial \eta}) = 0$ (8). We consider it possible to construct holomorphic functions $z_0(\zeta)$, $\omega_0(\zeta)$, $w_0(\zeta)$ efficiently. A solution of (7) (analogously, of (8)) is sought in the form $\varphi(\xi,\eta) = \sum\limits_{k=0}^{\infty}\varphi_k(\xi,\eta)$ (9), where $\varphi_0(\xi,\eta)$ (analogously, $\psi_0(\xi,\eta)$) is known, $\varphi_k(\xi,\eta)$ is defined by the Poisson formula $\varphi_k(\xi,\eta) = \iint_{\operatorname{Im}(\zeta)\geq 0} G(\xi,\eta;\xi_1,\eta_1)\varphi_{k-1}(\xi_{,1},\eta_1)d\xi_1 d\eta_1$, $k = \overline{1,\infty}$ (10), where G is Green's function for the half-plane with a coefficient. It remains to prove the convergence of series (9). Equation (8) is treated analogously.