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ON A TRACE INEQUALITY FOR ONE-SIDED POTENTIALS AND APPLICATIONS TO THE SOLVABILITY OF NONLINEAR INTEGRAL EQUATIONS

In the present note criteria governing the trace inequality

$$\int_0^\infty |R_\alpha f(x)|^p v(x)\,dx \le c_0 \int_0^\infty |f(x)|^p\,dx, \quad f \in L^p(R_+), \tag{1}$$

for the Riemann–Liouville integral operator

$$R_\alpha f(x) = \int_0^x f(y)(x-y)^{\alpha-1}\,dy, \quad x > 0,$$

are establish, where $0 < \alpha < 1/p$. Some applications to the solvability of a certain nonlinear integral equation are presented. The following statement holds:

Theorem 1. *Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. Then inequality (1) holds if and only if $W_\alpha v \in L^{p'}_{\mathrm{loc}}(R_+)$ and $W_\alpha[W_\alpha v]^{p'}(x) \le cW_\alpha v(x)$ a.e. where $p' = p/(p-1)$ and*

$$W_\alpha f(x) = \int_x^\infty f(y)(y-x)^{\alpha-1}\,dy.$$

An analogous problem for the Volterra integral operator

$$Kf(x) = \int_0^x f(y)k(x,y)\,d\mu(y)$$

is discussed.

Theorem 2. *Let $1 < p < \infty$, $0 < \alpha < 1$, and $A_p = (p'-1)(p')^{-p}$.*

(i) *If $W_\alpha v \in L^p_{\mathrm{loc}}(R_+)$ and the inequality $W_\alpha[W_\alpha v]^p(x) \le A_p W_\alpha v(x)$ a.e. holds, then the integral equation*

$$\varphi(x) = \int_x^\infty \frac{\varphi^p(t)}{(t-x)^{1-\alpha}}\,dt + \int_x^\infty \frac{v(t)}{(t-x)^{1-\alpha}}\,dt, \quad 0 < \alpha < 1 \tag{2}$$

has a non-negative solution $\varphi \in L^p_{\mathrm{loc}}(R_+)$. Moreover, $(W_\alpha v)(x) \le \varphi(x) \le p'(W_\alpha v)(x)$.

(ii) *If $0 < \alpha < 1/p'$ and (2) has a non-negative solution in $L^p_{\mathrm{loc}}(R_+)$, then $W_\alpha v \in L^p_{\mathrm{loc}}(R_+)$ and $W_\alpha[(W_\alpha v)^p](x) \le cW_\alpha v(x)$ a.e. for some constant $c > 0$.*