The one-variable fragment of a non-locally tabular modal logic can be finite

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A logic L is *k-tabular* if up to the equivalence in L, there exist only finitely many k-variable formulas.

L is *locally tabular* if it is k-tabular for all $k < \omega$.

Theorem (Maksimova, 1975)

For a logic $\mathrm{L} \supseteq \mathrm{S4}$, 1-tabularity implies local tabularity.

In other words:

For $\mathrm{L} \supseteq \mathrm{S4}$, if 1-generated free L-algebra is finite, then all finitely generated L-algebras are finite (i.e., the variety of L-algebras is locally finite).

Two questions (1970s)

- Does 1-tabularity imply local tabularity for every modal logic?
- Does 2-tabularity imply local tabularity for every intermediate logic?

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Does 1-tabularity imply local tabularity for every modal logic?

Does 2-tabularity imply local tabularity for every intermediate logic?

This talk:

- There exists a 1-tabular but not locally tabular modal logic
- k-tabularity, the top heavy property of canonical frames, and variants of Glivenko's theorem

Preliminaries

Language: a countable set $\textsc{Var}$ (propositional variables), Boolean connectives, a unary connective $\Diamond$ ($\Box$ abbreviates $\neg\Diamond\neg$).

Definition

A set of modal formulas L is a *normal modal logic* if L contains

- all tautologies
- $\Diamond\bot \leftrightarrow \bot, \quad \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$

and is closed under the rules of MP, substitution and monotonicity:
if $(\varphi \to \psi) \in \mathrm{L}$, then $(\Diamond\varphi \to \Diamond\psi) \in \mathrm{L}$.

Definition'

A set of modal formulas L is a *normal modal logic* if $\mathrm{L} = \{\varphi \mid A \vDash \varphi = \top\}$ for some modal algebra A.

TFAE:

L is *k-tabular*, i.e., up to the equivalence in L, there exist only finitely many k-variable formulas.	The free algebra $\mathfrak{A}_{\mathrm{L}}(k)$ is finite.	Every k-generated L-algebra is finite.

TFAE:

L is *locally tabular*, i.e., it is k-tabular for all $k < \omega$.	All $\mathfrak{A}_{\mathrm{L}}(k)$ are finite ($k < \omega$).	The variety of L-algebras is *locally finite*, i.e., every finitely generated L-algebra is finite.

L is *Kripke complete* if it is the logic of a class of frames.

L has the *finite model property* if it is the logic of a class of finite frames/algebras.

For every L,

$$\mathrm{L} = Log\{\mathfrak{A}_{\mathrm{L}}(k) \mid k < \omega\}.$$

L is locally tabular iff all $\mathfrak{A}_{\mathrm{L}}(k)$, $k < \omega$, are finite.

It follows that:

- If a logic is locally tabular, then it has the finite model property (thus, it is Kripke complete).
- Every extension of a locally tabular logic is locally tabular (thus, it has the finite model property).
- Every finitely axiomatizable extension of a locally tabular logic is decidable.

Some locally tabular modal logics (locally finite varieties of modal algebras)

$$B_0 = \bot, \quad B_1 = p_1 \to \Box\Diamond p_1, \quad B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$$

(Segerberg, 1971) B_h is valid in a preorder F iff the height of $\mathrm{F} \leq h$.

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For a logic $\mathrm{L} \supseteq \mathrm{S4}$, TFAE:

- L is locally tabular
- L is of finite height, i.e., contains some B_h
- L is the logic of a class $\mathcal{F}$ of preorders s.t. $\exists h < \omega \; \forall \mathrm{F} \in \mathcal{F} \;\; ht(\mathrm{F}) \leq h$
- L is 1-tabular

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(Nagle, 1981; Nagle, Thomason, 1985) $\mathrm{K5} = [\Diamond p \to \Box\Diamond p]$ is locally tabular.
This logic is non-transitive. It is a *2-transitive logic of height 2*.

(Gabbay, Shehtman, 1998; Shehtman, 2014). $\mathrm{K}_n + \Box^s\bot$ is locally tabular ($n > 0$, $\Box^s$ is a non-empty sequence of boxes).

(N. Bezhanishvili, 2002) Every proper extension of $\mathrm{S5} \times \mathrm{S5}$ is locally tabular.

(Shehtman, Sh, 2016) The criterion of Segerberg and Maksimova holds for extensions of logics much weaker than $\mathrm{S4}$. In particular, it holds if, for some $m \geq 2$, L contains

$$\underbrace{\Diamond \ldots \Diamond}_{m \text{ times}} p \to \Diamond p \vee p$$

Part 1. There exists a 1-tabular but not locally tabular modal logic.

What are 1-tabular logics?

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(Shehtman, Sh)

If L is 1-tabular, then

- L is *pretransitive*, and
- L is of *finite height*.

A logic L is *pretransitive* if there exists a one-variable formula $\Diamond^*(p)$ ('master modality') s.t. L contains

$$\Diamond^*(\Diamond^*(p)) \to \Diamond^*(p), \quad p \to \Diamond^*(p), \quad \text{and} \quad \Diamond p \to \Diamond^*(p).$$

Put $\Box^*\varphi = \neg\Diamond^*(\neg\varphi)$. At a point of a model of L it expresses the trues of φ 'everywhere in the point-generated submodel'.

Synonyms: EDPC-logics (Blok and Pigozzi), logics with expressible master modality (Kracht), conically expressive logics (Shehtman).

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Theorem (Kowalski and Kracht, 2006) L is pretransitive iff L is *m-transitive* for some $m \geq 0$, i.e., contains $\Diamond^{m+1}p \to p \vee \Diamond p \vee \ldots \vee \Diamond^m p$

This means that the 'master modality' operator $\Diamond^*\varphi$ is always of form
$\varphi \vee \Diamond\varphi \vee \ldots \vee \Diamond^m\varphi$
(The same it true in the polymodal language $\Diamond_1, \ldots, \Diamond_n$: write $\Diamond p$ for $\vee\Diamond_i p$.)

In Kripke semantics, the formula of m-transitivity says

"if y is accessible from x in $m+1$ steps, then y is accessible from x in $\leq m$ steps"

Some pretransitive examples

$\mathrm{K4}, \mathrm{wK4} = [\Diamond\Diamond p \to \Diamond p \vee p]$	1-transitive
$\mathrm{K5} = [\Diamond p \to \Box\Diamond p]$	2-transitive
$[\Diamond^n p \to \Diamond^m p]$, $n > m$	$(n-1)$-transitive
$[\neg\Diamond^m\top]$, $m > 0$	(m-1)-transitive
The (expanding) product of two transitive logics	2-transitive

L is pretransitive iff L contains $\Diamond^{m+1}p \to p \vee \Diamond p \vee \ldots \vee \Diamond^m p$

Another pretransitive example

The logic of a finite frame (tabular logic) is pretransitive.

$$\mathrm{L} \text{ is 1-tabular} \quad \Rightarrow \quad \mathrm{L} \text{ is pretransitive.}$$

Proof.

Consider the 1-generated canonical frame of L.
This frame is finite.
Thus, it validates some *m*-transitivity formula.
This formula is one-variable, thus L contains it. □

Frames of finite height

A poset F is of *finite height* $\leq h$ if its every chain contains at most h elements.

R^* denotes the transitive reflexive closure of R:

$$R^* = Id \cup R \cup R^2 \cup \ldots$$

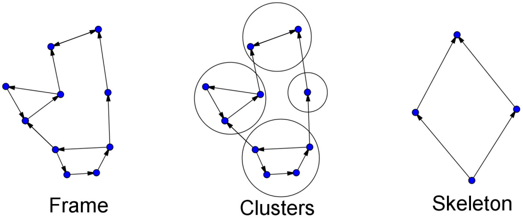
An equivalence class w.r.t. $\sim_R = R^* \cap R^{*\,-1}$ is called a *cluster* (so clusters are maximal subsets where R^* is universal).

The *skeleton of (W, R)* is the poset $(W/{\sim_R}, \leq_R)$, where for clusters C, D,

$$C \leq_R D \qquad \text{iff} \qquad x\, R^* y \text{ for some } x \in C,\ y \in D.$$

Height of a frame is the height of its skeleton.

Remark: In the polymodal case, the height of $(W, R_1, \ldots, R_n)$ is the height of $(W, \cup R_i)$.



$$B_0 = \bot, \quad B_1 = p_1 \to \Box\Diamond p_1, \quad B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$$

A pretransitive logic is *of finite height* if it contains a formula B_h^* for some h, where B_h^* is obtained from B_h by replacing $\Diamond$ with $\Diamond^*$ and $\Box$ with $\Box^*$

Proposition. For a pretransitive frame F, $\quad \mathrm{F} \models B_h^* \quad \text{iff} \quad ht(\mathrm{F}) \leq h$.

Examples

$\mathrm{S5}$: height=1
$\mathrm{K5}$: height=2
$\mathrm{S5} \times \mathrm{S5}$: height=1

L is 1-tabular $\Rightarrow$ L is of finite height.

Proof.

The $*$-fragment $^*\mathrm{L}$ of L is a logic containing $\mathrm{S4}$.
If L is 1-tabular, then $^*\mathrm{L}$ is.
Then $^*\mathrm{L}$ is locally tabular (Maksimova's theorem).
Then $^*\mathrm{L}$ is of finite height (Maksimova and Segerberg criterion).
Thus, L contains some B_h^*. $\Box$

If L is 1-tabular, then for some m, h, L is the logic of a class of m-transitive frames of height $\leq h$.

In general, pretransitive logics of height 1 are not locally tabular (and not 1-tabular):

The logic of reflexive symmetric frames (W, R) such that

$$R \circ R = W \times W$$

is not locally tabular (Byrd, 1978).

Moreover, its one-variable fragment is infinite (Makinson, 1981).

This logic is 2-transitive; its height is 1.

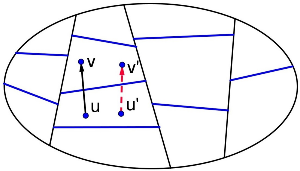
Locally tabular logics are Kripke complete.

What can we say about their frames?

Let $\mathrm{F} = (W, R)$ be a frame. A partition $\mathcal{A}$ of W is *tuned* if for every $U, V \in \mathcal{A}$,

$$\exists u \in U \; \exists v \in V \; uRv \quad \Rightarrow \quad \forall u \in U \; \exists v \in V \; uRv.$$

F is said to be *tunable* if every finite partition $\mathcal{A}$ of F admits a tuned finite refinement $\mathcal{B}$.



Proposition (Franzen, Fine, 1970s)

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For a finite partition $\mathcal{B}$ of W,
$\mathcal{B}$ is tuned iff $\{\cup x \mid x \subseteq \mathcal{B}\}$ forms a subalgebra of $(\mathcal{P}(W), R^{-1})$. $\square$

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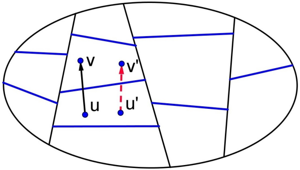

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Proof. $(\omega, \leq)$ is tunable (this is a very simple exercise: refine $\mathcal{A}$ in such a way that all elements of $\mathcal{B}$ are infinite or singletons, and singletons cover an initial segment of ω).

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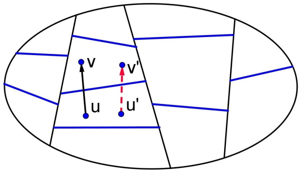

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A spinoff

Let $(\omega^n, \preceq)$ be the direct product of $n < \omega$ instances of $(\omega, \leq)$

Theorem. For all finite n, $(\omega^n, \preceq)$ has the FMP.

Proof. $(\omega^n, \preceq)$ is tunable (more entertaining exercise...).

Question. Is $(\omega^n, \preceq)$ finitely axiomatizable for $n > 1$? Is it decidable?

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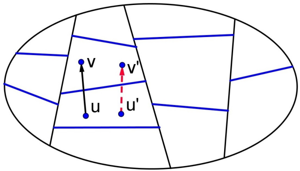

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Proof. $(\omega, \leq)$ is tunable (this is a very simple exercise: refine $\mathcal{A}$ in such a way that all elements of $\mathcal{B}$ are infinite or singletons, and singletons cover an initial segment of ω).

While the algebra of $(\omega, \leq)$ is locally finite, $Log(\omega, \leq)$ is not locally tabular: $(\omega, \leq)$ is of infinite height.
A hint: the size of $\mathcal{B}$ can be arbitrary large even for the case $|\mathcal{A}| = 2$.

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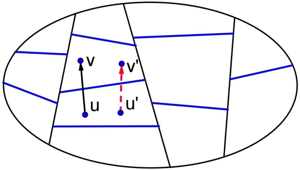

A class of frames $\mathcal{F}$ is *ripe*, if there exists $f : \omega \to \omega$ s.t. for every finite partition $\mathcal{A}$ of a frame $\mathrm{F} \in \mathcal{F}$ there exists a tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

Theorem (Shehtman, Sh)

The following are equivalent:

(1) L is locally tabular.

(2) L is the logic of a ripe class of frames.

(3) L is a Kripke complete pretransitive logic of finite height and $\mathit{Clust}(\mathrm{L})$ is ripe.

Here $\mathit{Clust}(\mathrm{L})$ is the class of clusters occurring in L-frames:
$\mathit{Clust}(\mathrm{L}) = \{\mathrm{F}{\restriction}C \mid \mathrm{F} \vDash \mathrm{L} \text{ and } C \text{ is a cluster in } \mathrm{F}\}$.

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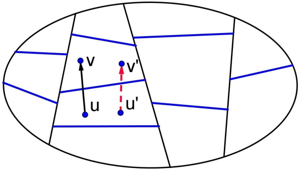

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Theorem (G. Bezhanishvili, 2001)

If, for all finite k, the size of all k-generated algebras in a class of algebras is uniformly bounded by a finite $n(k)$, then the variety generated by this class is locally finite.

Since tuned partitions relate to subalgebras, (1) $\iff$ (2) can be obtained as a corollary.

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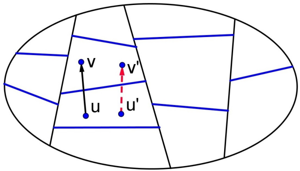

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Theorem (Segerberg, Maksimova, 1970s). For a logic $\mathrm{L} \supseteq \mathrm{S4}$,

$$\mathrm{L} \text{ is locally tabular iff it is of finite height.}$$

Proof If (W, R) is a cluster in a preorder, then $R = W \times W$ and so any partition of (W, R) is tuned.

(If you do not like tuned partitions, there is another explanation: all $\mathrm{S4} + B_h$ are locally tabular, because $\mathrm{S5}$ is locally tabular.)

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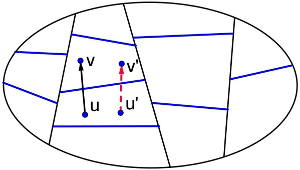

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Corollary. If $\mathit{Clust}(\mathrm{L}_0)$ is ripe and L_0 is pretransitive and canonical, then for any $\mathrm{L} \supseteq \mathrm{L}_0$, TFAE:

- L is locally tabular
- L is of finite height
- L is 1-tabular

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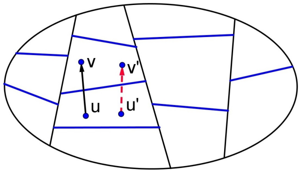

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Corollary. For L containing $\Diamond^m p \to \Diamond p \vee p$ $(m \geq 2)$, TFAE:

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- L is of finite height
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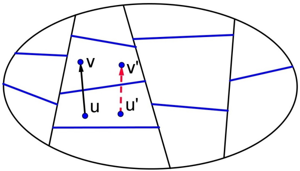

A class of frames $\mathcal{F}$ is *ripe*, if there exists $f : \omega \to \omega$ s.t. for every finite partition $\mathcal{A}$ of a frame $\mathrm{F} \in \mathcal{F}$ there exists a tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

Theorem (Shehtman, Sh)

The following are equivalent:

(1) L is locally tabular.

(2) L is the logic of a ripe class of frames.

(3) L is a Kripke complete pretransitive logic of finite height and $Clust(\mathrm{L})$ is ripe.

Here $Clust(\mathrm{L})$ is the class of clusters occurring in L-frames:
$Clust(\mathrm{L}) = \{\mathrm{F}{\restriction}C \mid \mathrm{F} \vDash \mathrm{L} \text{ and } C \text{ is a cluster in } \mathrm{F}\}$.

Corollary. For L containing $\Diamond^m p \to \Diamond p \vee p$ $(m \geq 2)$, TFAE:

- L is locally tabular
- L is of finite height
- L is 1-tabular

Let $\mathrm{F} = (W, R)$ be a frame. A partition $\mathcal{A}$ of W is *tuned* if for every $U, V \in \mathcal{A}$,

$$\exists u \in U \ \exists v \in V \ uRv \quad \Rightarrow \quad \forall u \in U \ \exists v \in V \ uRv.$$

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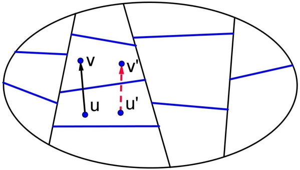

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Weird corollary...

If the logic of a frame is locally tabular, then the logic of any its subframe is locally tabular.

What are locally tabular modal logics?

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What are locally tabular pretransitive logics of height 1?

Or:

What are locally tabular logics of pretransitive clusters?

Or:

What are ripe clusters?

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$$\mathrm{F} = (\omega + 1, R),$$

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$$xRy \qquad \text{iff} \qquad x \leq y \text{ or } x = \omega.$$

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The logic of F is one-tabular but not locally tabular.

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The logic of F *is one-tabular but not locally tabular.*

Proof.

1-tabularity:

The logic of a frame is one-tabular iff there exists $c \in \omega$ such that for every two-element partition $\mathcal{A}$ there exists its tuned refinement $\mathcal{B}$ with $|\mathcal{B}| \leq c$.

If $\{U_0, U_1\}$ is a two-element partition of $\omega + 1$, then there exists its tuned refinement $\mathcal{B}$ with $|\mathcal{B}| \leq 3$. Thus, the logic of F is one-tabular.

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Non-local finiteness: The restriction of $(\omega + 1, R)$ onto ω is the frame $(\omega, \leq)$, which is of infinite height. Thus $Log\,(\omega + 1, R)$ is not locally tabular (by Weird Corollary).

□

Problems

1. Does k-tabularity imply local tabularity, for some fixed k for all modal logics? For $k = 2$?
2. The same questions for intermediate logics.

Possible reformulation of Question 1

Suppose that we can "tune" three-element partitions of a cluster, i.e.

there exists c s.t. for every 3-element partition $\mathcal{A}$ there exists its tuned refinement $\mathcal{B}$ with $|\mathcal{B}| < c$.

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Can we "tune" all finite ones?

When we can tune 3-element partitions, every subframe is of finite height...

Part 2. k-tabularity, the top heavy property of canonical frames, and variants of Glivenko's theorem

Glivenko's theorem (h=1)

Formulas of finite height

modal: $B_0 = \bot$, $B_1 = p_1 \to \Box\Diamond p_1$, $B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$

intermediate: $B'_0 = \bot$, $B'_1 = p_1 \vee (p_1 \to \bot)$, $B'_{i+1} = p_{i+1} \vee (p_{i+1} \to B'_i)$

$\mathrm{L}[h]$ is the extension of L with the formula of height h.

$$\mathrm{CL} = \textsc{Int}[1], \quad \mathrm{S5} = \mathrm{S4}[1]$$

Theorem (Glivenko, 1929) $\mathrm{CL} \vdash \varphi$ iff $\textsc{Int} \vdash \neg\neg\varphi$.

Proof. $\textsc{Int}$ has the FMP: it is the logic of posets. Every point in a finite poset sees a model of CL — a maximal point. □

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Theorem (Kudinov, Sh) For all pretransitive L, $\mathrm{L}[1] \vdash \varphi$ iff $\mathrm{L} \vdash \Diamond^*\Box^*\varphi$.

Proof. The Esakia-Fine *maximality lemma* holds in pretransitive canonical frames. □

Maximality lemma in pretransitive canonical frames

For $k \leq \omega$, the *k-canonical frame* (W, R) of L is built from maximal L-consistent sets of k-formulas (in variables p_i, $i < k$);
$xRy \quad \text{iff} \quad \{\Diamond\varphi \mid \varphi \in y\} \subseteq x$.
For a k-formula φ, put $\|\varphi\| = \{x \in W \mid \varphi \in x\}$.

$$\varphi \in \mathrm{L} \qquad \text{iff} \qquad \|\varphi\| = W$$

Lemma

In the pretransitive case, if $\varphi \in x \in W$, then $R^*(x) \cap \|\varphi\|$ has a maximal (w.r.t. the preorder R^*) element.

Proof.

For $y \in W$,

$$R^*(y) = \bigcap\{\|\alpha\| \mid \Box^*\alpha \in y\};$$

thus the set $R^*(y) \cap \|\varphi\|$ is closed in the Stone topology on W.
By the compactness, $\bigcap\{R^*(y) \cap \|\varphi\| \mid y \in \Sigma\}$ is non-empty for any R^*-chain Σ in $R^*(x) \cap \|\varphi\|$; thus Σ has an upper bound in $\|\varphi\|$. □

Corollary

In the pretransitive case, $\quad \mathrm{L}[1] \vdash \varphi \quad$ *iff* $\quad \mathrm{L} \vdash \Diamond^*\Box^*\varphi$.

Proof. Put $k = \omega$, $\varphi = \top$. Every point in W sees (via R^*) a model of $\mathrm{L}[1]$ — an R^*-maximal cluster.

Glivenko's theorem ($h < \omega$)

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$$
\begin{array}{lcl}
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$$\mathrm{L}[3] \vdash \varphi \qquad \text{iff} \qquad \mathrm{L} \vdash ???$$
$$\ldots$$

If $\mathrm{L}[h]$ is k-tabular, then there exists a Glivenko-type translation from $\mathrm{L}[h+1]$ to L for k-formulas:

$$\mathrm{L}[h+1] \vdash \varphi \qquad \text{iff} \qquad \mathrm{L} \vdash \mathrm{TR}_{h,k}(\varphi)$$

The proof is based on the *top-heavy* property of finitely generated pretransitive canonical frames.

The *depth* of x in a frame $\mathrm{F} = (W, R)$ is the height of the point-generated frame $\mathrm{F}[x]$.
$W[\leq h]$ is the set of points of depth $\leq h$.

F is *h-heavy* if every its element x which is not in $W[\leq h]$ sees a point of depth h.
F is *top-heavy*, if it is h-heavy for all finite $h > 0$.

Theorem (Shehtman, 1979)

All finitely generated canonical $\mathrm{S4}$*-frames are top-heavy.*

Historical remarks

(Esakia, Grigolia, 1975):
Description of 1-generated canonical frames for $\mathrm{S4.3}$ and $\mathrm{Grz.3}$

The term ‘top-heavy’ is due to (Fine, 1985)

The *depth* of x in a frame $\mathrm{F} = (W, R)$ is the height of the point-generated frame $\mathrm{F}[x]$. $W[\leq h]$ is the set of points of depth $\leq h$. $\mathrm{F}[\leq h]$ is the restriction of F on $W[\leq h]$.

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Theorem

Let $\mathrm{F} = (W, R)$ *be the* k*-canonical frame of* L, $\mathrm{L}[h]$ *be* k*-tabular* $(h, k < \omega)$.

1. *For* $i \leq h$, *the set* $W[\leq i]$ *is definable in* F: *there is a formula* $\mathbf{B_i}$ *such that*

$$\mathbf{B_i} \in x \qquad \textit{iff} \qquad x \in W[\leq i].$$

2. F *is* $(h+1)$*-heavy.*
3. *For all* k*-formulas* φ,

$$\mathrm{L}[h+1] \vdash \varphi \qquad \textit{iff} \qquad \mathrm{L} \vdash \mathbf{C_0}(\varphi) \wedge \ldots \wedge \mathbf{C_h}(\varphi),$$

where $\mathbf{C_i}(\varphi) = \Box^*(\Box^*\varphi \to \mathbf{B_i}) \to \mathbf{B_i}$.

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where $\mathbf{C_i}(\varphi) = \Box^*(\Box^*\varphi \to \mathbf{B_i}) \to \mathbf{B_i}$.

1: For a in $\mathrm{F}[\leq h]$, let $\alpha(a)$ define a in $\mathrm{F}[\leq h]$.

Then a is defined in F by the conjunction of $\alpha(a)$ with the following frame-like formula

$\Box^* \bigwedge \{\alpha(b_1) \to \Diamond\alpha(b_2) \mid b_1, b_2 \in W[\leq h],\ (b_1, b_2) \in R\} \quad \wedge$

$\Box^* \bigwedge \{\alpha(b_1) \to \neg\Diamond\alpha(b_2) \mid b_1, b_2 \in W[\leq h],\ (b_1, b_2) \notin R\} \quad \wedge$

$\Box^* \vee \{\alpha(b) \mid b \in W[\leq h]\}$

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where $\mathbf{C_i}(\varphi) = \Box^*(\Box^*\varphi \to \mathbf{B_i}) \to \mathbf{B_i}$.

2: Follows from (1) and the maximality lemma: if x is not in $W[\leq h]$, then there exists a maximal y in $R^*(x) \setminus W[\leq h]$. The depth of y in F is $h+1$, as required.

3: Straightforward from (2).

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where $\mathbf{C_i}(\varphi) = \Box^*(\Box^*\varphi \to \mathbf{B_i}) \to \mathbf{B_i}$.

$$\mathrm{S5} \vdash \varphi \qquad \text{iff} \qquad \mathrm{S4} \vdash \Diamond\Box\varphi$$

The strangest explanation: the inconsistent logic $\mathrm{S4}[0]$ is k-tabular for all k,

$\mathbf{B_0}$ is always $\bot$, and $\mathbf{C_0}(\varphi)$ is $\Box(\Box\varphi \to \bot) \to \bot$.

We know that all $\mathrm{S4}[h]$ are locally tabular.

Thus, for all finite k, h there exists a formula $\mathrm{TR}_{h,k}(\mathbf{s})$ in variables p_i, $i < k$ and $\mathbf{s}$, s.t. for any k-variable φ

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$$\mathrm{S4}[h+1] \vdash \varphi \qquad \text{iff} \qquad \mathrm{S4} \vdash \mathrm{TR}_{h,k}(\varphi)$$

Theorem (Kuznetsov, 1971; Komori,1975)

All $\mathrm{INT}[h]$ *are locally tabular.*

Theorem (Shehtman, 1983)

All finitely generated canonical $\mathrm{INT}[h]$*-frames are top-heavy.*

Likewise, for all finite k, h there exists $\mathrm{TR}_{h,k}(\varphi)$ s.t. for any k-variable φ

$$\mathrm{INT}[h+1] \vdash \varphi \qquad \text{iff} \qquad \mathrm{INT} \vdash \mathrm{TR}_{h,k}(\varphi)$$

Remark

In these cases, $\mathrm{TR}_{h,k}$ can be effectively constructed from k and h.

Concluding remarks

Finite height is not a necessary condition for local tabularity of intermediate logics.

What can an analog of Gliveko's translation be in the case of a locally tabular intermediate logic with no finite height axioms?

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What can an analog of Gliveko's translation be in the case of a locally tabular intermediate logic with no finite height axioms?

MIPC, or $\mathrm{IS5}$ (the logic of *monadic Heyting algebras*):

$$\begin{array}{ll}
\textsc{Int} & \\
\Box p \to p & p \to \Diamond p \\
(\Box p \wedge \Box q) \to \Box(p \wedge q) & \Diamond(p \vee q) \to \Diamond p \vee \Diamond q \\
\Diamond p \to \Box\Diamond p & \Diamond\Box p \to \Box p \\
\Box(p \to q) \to (\Diamond p \to \Diamond q) & \\
\text{Rules: MP, Sub, necessitation} &
\end{array}$$

(G. Bezhanishvili, 2001): For all L between MIPC and $\mathrm{WS5} = \mathrm{MIPC} + \Diamond p \leftrightarrow \neg\Box\neg p$, we have $\mathrm{WS5} \vdash \varphi$ iff $\mathrm{L} \vdash \neg\neg\Box\varphi$.

(G. Bezhanishvili, R.Grigolia, 1998): Locally tabular extensions of MIPC.

Can we use these results on local finiteness to obtain 'finite height' variants of Glivenko's theorem in the context of intuitionistic modal logic?

Thank you!